Sequences of Orthogonal Polynomials with Step Functions as Their Weight Functions

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Abstract: In this paper, we will construct the sequences of orthogonal polynomials with step functions as their weight functions in the interval [0,1]. We will show these polynomials up to degree seven. The application of these orthogonal polynomials will be illustrated with example.

Key-Words: Orthogonal-polynomial Step-function Weight-function Gauss-Quadrature

1. Introduction

The orthogonal polynomial of degree n on the interval [0,1] with respect to the weight function $w(x)$ is in the form

$p_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ with the property that

$$\int_0^1 w(x)p_n(x)q_m(x)dx = 0$$

for all polynomials $q_m(x)$ of degree $m \le n-1$.

The sequence of orthogonal polynomials $\{p_0(x), p_1(x), p_2(x), ..., p_n(x), ...\}$ is said to be the sequence of orthogonal polynomials on the interval [0,1] with respect to the weight function $w(x)$ if all of orthogonal polynomials $p_k(x)$'s are the orthogonal polynomials on the interval [0,1] with respect to the weight function $w(x)$. In this paper, We will introduce six sequences of orthogonal polynomials with step functions as their weight functions.

2. Formulation

The six sequences of orthogonal polynomials are as follow.

First Sequence

Let $w(x) = \begin{cases} 0, & 0 \le x < 0.5 \\ 2, & 0.5 \le x \le 1 \end{cases}$.

The first eight orthogonal polynomials are;

$p_0(x) = 1$

$p_1(x) = x - 0.5$

$p_2(x) = x^2 - 0.5416666667x + 1.5$

$p_3(x) = x^3 - 2.25x^2 + 1.65x - 0.3975$

$p_4(x) = x^4 - 3x^3 + 3.3214285714x^2 - $
$-1.6071428571x + 0.2866071429$

$p_5(x) = x^5 - 3.75x^4 + 5.5555555556x^3 - $
$4.0625x^2 + 1.4657738095x - 0.20870535714$

$p_6(x) = x^6 - 4.5x^5 + 8.3522727272x^4 - $
$8.1818181818x^3 + 4.4602272727x^2 - $
$1.2826704545x + 0.1520055465$

$p_7(x) = x^7 - 5.25x^6 + 11.7115384615x^5 - $
$14.3870192308x^4 + 10.5091783217x^3 - $
$4.5639204545x^2 + 1.0909455128x - $
0.1107203344

Second Sequence

Let $w(x) = \begin{cases} 2, & 0 \le x < 0.5 \\ 0, & 0.5 \le x \le 1 \end{cases}$

The first eight orthogonal polynomials are;

$p_0(x)=1$

$p_1(x)=x-0.5$

$p_2(x)=x^2-0.5x+0.0416666667$

$p_3(x)=x^3-0.75x^2+0.15x-0.00625$

$p_4(x)=x^4-x^3+0.3214285714x^2-0.35714285714x+0.00089185714$

$p_5(x)=x^5-1.25x^4+0.5555555556x^3-0.1041666667x^2+0.7440761905x-0.0001240079$

$p_6(x)=x^6-1.5x^5+08522727272x^4-0.0227272727x^3+0.0284090909x^2-0.0014204545x+0.0000169102$

$p_7(x)=x^7-1.75x^6+1.2115384615x^5-0.4206730769x^4+0.0764860140x^3-0.0068837413x^2+0.0002549534x-0.0000022764$

Third Sequence

Let $w(x)=\begin{cases}0, & 0\le x<1/3\\ 3, & 1/3\le x<2/3\\ 0, & 2/3\le x\le 1\end{cases}$.

The first eight orthogonal polynomials are;

$p_0(x)=1$

$p_1(x)=x-0.5$

$p_2(x)=x^2-x+0.2407407407$

$p_3(x)=x^3-1.5x^2+0.7333333333x-0.1166666667$

$p_4(x)=x^4-2x^3+1.4761904762x^2-0.4761904762x+0.0566137566$

$p_5(x)=x^5-2.5x^4+2.4691358025x^3-1.2037037037x^2+0.2895355673x-0.0274838330$

$p_6(x)=x^6-3x^5+3.7121212121x^4-2.4242424242x^3+0.8810325477x^2-0.1689113356x+0.0133447942$

$p_7(x)=x^7-3.5x^6+5.2051282051x^5-4.2628205128x^4+2.0758870759x^3-0.6010101010x^2+0.0957757378x-0.0064802025$.

Forth Sequence

Let $w(x)=\begin{cases}2/3, & 0\le x<0.5\\ 4/3, & 0.5\le x\le 1\end{cases}$.

The first eight orthogonal polynomials are;

$p_0(x)=1$

$p_1(x)=x-0.5$

$p_2(x)=x^2-1.0454545455x+0.1931818181$

$p_3(x)=x^3-1.5628654971x^2+0.6634502924x-0.0587353801$

$p_4(x)=x^4-2.0504780575x^3+1.3641869893x^2-0.3181179296x+0.0167910801$

$p_5(x)=x^5-2.5592610403x^4+2.3415924271x^3-0.9092894413x^2+0.1349605699x-0.0046757078$

$p_6(x)=x^6-3.0522286626x^5+3.5420637705x^4-1.9382084065x^3+0.4997751196x^2-0.0517503342x+0.0012780051$

$p_7(x)=x^7-3.5578364339x^6+5.0206520786x^5-3.5636670512x^4+1.3283826114x^3-0.2456620273x^2+0.0018719225x-0.0003439212$.

Fifth Sequence

Let

$w(x)=\begin{cases}1.5, & 0\le x<1/3\\ 0, & 1/3\le x<2/3\\ 1.5, & 2/3\le x\le 1\end{cases}$.

The first eight orthogonal polynomials are;

$p_0(x)=1$

$p_1(x)=x-0.5$

$p_2(x)=x^2-x+0.1296296296$

$p_3(x)=x^3-1.5x^2+0.5948717949x-0.0474358974$

$p_4(x)=x^4-2x^3+1.2373145980x^2-0.2373145980x+0.00977534912$

$p_5(x)=x^5-2.5x^4+2.2070137021x^3-0.8105205532x^2+0.1104074998x-0.0034503244$

$p_6(x) = x^6 - 3x^5 + 3.3477440056x^4 - 1.6954881713x^3 + 0.3804668848x^2 - 0.0327227992x + 0.0006570135$

$p_7(x) = x^7 - 3.5x^6 + 4.8187842514x^5 - 3.2969606129x^4 + 1.1642962057x^3 - 0.1994836956x^2 + 0.1382102954x - 0.00022859$.

Sixth Sequence

Let

$$w(x) = \begin{cases} 0.5, & 0 \le x < 1/3 \\ 1, & 1/3 \le x < 2/3 \\ 1.5, & 2/3 \le x \le 1 \end{cases}.$$

The first eight orthogonal polynomials are;

$p_0(x) = 1$

$p_1(x) = x - 0.5$

$p_2(x) = x^2 - 1.0869565217x + 0.2198067633$

$p_3(x) = x^3 - 1.5750643692x^2 + 0.6796593384x - 0.0640841971$

$p_4(x) = x^4 - 2.0909736824x^3 + 1.4270796895x^2 - 0.3438897949x + 0.0187368216$

$p_5(x) = x^5 - 2.5777446380x^4 + 2.3844400859x^3 - 0.9433146908x^2 + 0.1450368782x - 0.0052456985$

$p_6(x) = x^6 - 3.0836348359x^5 + 3.6222647243x^4 - 2.0110920566x^3 + 0.5277334666x^2 - 0.0558422318x + 0.0014127870$

$p_7(x) = x^7 - 3.5845979575x^6 + 5.1059549776x^5 - 3.6682204298x^4 + 1.3895340944x^3 - 0.2625834635x^2 + 0.0205263560x - 0.0003872573$.

3. Gauss-Quadrature Formula

Gauss-Quadrature formula is the formula for finding the aprximated value of the definite integral which is in the form

$$\int_a^b w(x)f(x)dx \cong \sum_{k=1}^{n} A_k f(x_k)$$

where x_k's are the roots of the orthogonal of degree n, $p_n(x)$, on the interval $[a,b]$ with respect to the weight function $w(x)$ and

$$A_k = \frac{1}{p_n'(x_k)} \int_a^b \frac{w(x)p_n(x)}{x - x_k} dx.$$

The following tables are the table of the points x_k's and the values A_k's of the above orthogonal polynomials.

The following tables are the table of the points a_k's and the values A_k's of the above orthogonal polynomials.

n	x_k	A_k
1	0.5000000000	1.0000000000
2	0.6056624327 0.8943375673	1.3660254038 -0.3660254038
3	0.5563508326 0.7500000000 0.9436491673	-2.8888888889 2.5899416688 1.2989472201
4	0.5347159222 0.6650047391 0.8349952609 0.9652840779	6.3020742435 -11.2763732373 8.7334701401 -2.7591711463
5	0.5234550385 0.6153826725 0.7500000000 0.8846173275 0.9765449615	18.10085195710 -40.68870622427 42.3511111111 -26.07951658671 8.3162597428
6	0.5168826214 0.5846976534 0.6003452035 0.8096547965 0.9153023466 0.9831173786	58.3297359639 -149.1381890430 179.4016159134 -148.5546808270 87.7716993838 -26.1018139120
7	0.5127230219 0.5646172036 0.6485387122 0.7500000000 0.8514612878 0.9353827964 0.9872769781	60.6746933907 -126.6055373802 88.72545731010 -0.0083393184 -57.4598086256 56.9648410778 -21.2913064545

Table 1 $w(x) = \begin{cases} 0, & 0 \le x < 0.5 \\ 2, & 0.5 \le x \le 1 \end{cases}$

n	x_k	A_k
1	0.5000000000	1.0000000000
2	0.1056624327	-0.3660254038
	0.3943375673	1.3600254038
3	0.0563508327	1.2989472201
	0.2500000000	-2.8888888889
	0.4436491673	2.5899416688
4	0.0347159221	-2.7591711463
	0.1650047391	8.7334701401
	0.3349952609	-11.2763732373
	0.4652840779	6.3020742435
5	0.0234550385	8.3162597428
	0.1153826725	-26.0795165867
	0.2500000000	41.3511111111
	0.3846173275	-40.6887062243
	0.4765449615	18.1008519571
6	0.0168826214	-26.8101813912
	0.0846976534	87.7716993838
	0.1903452035	-148.5546808270
	0.3096547965	179.4016159134
	0.4153023466	-149.1481890430
	0.4831173786	58.3297359639
7	0.0127230222	94.8688557837
	0.0646172036	-315.0890432623
	0.1485387122	557.1221084993
	0.2450000000	-736.2481632653
	0.3514612878	762.0132897338
	0.4353827996	-566.0528307570
	0.4872769781	204.3857832742

Table 2 $w(x)=\begin{cases}2, & 0\le x<0.5\\ 0, & 0.5\le x\le 1\end{cases}$

n	x_k	A_k
1	0.5000000000	1.0000000000
2	0.4037749551	0.5000000000
	0.5962250449	0.5000000000
3	0.3709005551	2.5000000000
	0.5000000000	-4.0000000000
	0.6290994449	2.5000000000
4	0.3564772814	2.3039595906
	0.4433364927	-1.8039595906
	0.5566635073	-1.8039595906
	0.6435227186	2.3039595906
5	0.3489700257	17.5719199857
	0.4102551150	-44.5919199857
	0.4500000000	55.0400000000
	0.5897448850	-44.5919199857
	0.6510299743	17.5719199857
6	0.3445884143	20.9844822998
	0.3897984356	-40.9711200349
	0.4602301357	20.4866377352
	0.5397698643	20.4866377352
	0.6102015644	-40.9711200349
	0.6554115857	20.9844822998
7	0.3418153479	-2.3527156680
	0.3764114691	-0.6179878514
	0.4323591414	43.1263903414
	0.4500000000	-133.1770998347
	0.5676408586	214.2579139544
	0.6235885309	-202.4650336923
	0.6581846521	82.2285327506

Table 3

$$w(x)=\begin{cases}0, & 0\le x<1/3\\ 3, & 1/3\le x<2/3\\ 0, & 2/3\le x\le 1\end{cases}$$

n	x_k	A_k
1	0.5000000000	1.0000000000
2	0.2397750091	0.3929040895
	0.8056795363	0.6070959105
3	0.1196945387	0.1971953135
	0.5484526011	0.4578967348
	0.8947183573	0.3449079518
4	0.0734604906	0.1229528595
	0.3563477130	0.2504895984
	0.6870045536	0.4052338144
	0.9336653003	0.2213237277
5	0.0487540728	0.0821315924
	0.2406970070	0.1675043239
	0.5350902760	0.2966878418
	0.7796888263	0.3023707147
	0.9550308580	0.1513055272
6	0.0350439728	0.0593050052
	0.1763606851	0.1258622892
	0.4030783272	0.1791493257
	0.6342390093	0.2931856689
	0.8362106516	0.2319082641
	0.9672960166	0.1105894468
7	0.0261805153	0.0444182826
	0.1331241008	0.0962145122
	0.3071142571	0.1328576507
	0.5278474413	0.2196360852
	0.7134622225	0.2429426819
	0.8748326868	0.1802335192
	0.9753212103	0.0836972682

Table 4

$$w(x)=\begin{cases}2/3, & 0\le x<0.5\\ 4/3, & 0.5\le x\le 1\end{cases}$$

n	x_k	A_k
1	0.5000000000	1.0000000000
2	0.1530556668 0.8469443332	0.5000000000 0.5000000000
3	0.1061368193 0.5000000000 0.8938631807	0.2685950413 0.4628099154 0.2685950413
4	0.0562104723 0.2436091712 0.7563908288 0.9437895277	0.0670554414 0.4329445586 0.4329445586 0.0670554414
5	0.0436637199 0.2088926361 0.4500000000 0.7911073639 0.9563362801	0.1057250867 0.2318783050 0.3247932165 0.2318783050 0.1057250867
6	0.0282556233 0.1376506980 0.2799633195 0.7200366848 0.8623493020 0.9717443767	0.1221274604 -0.0459339394 0.4238064790 0.4238064790 -0.0459339394 0.1221274604
7	0.0234707223 0.1177912100 0.2590600838 0.5000000000 0.7409399162 0.8822087900 0.9765292777	0.0622653676 0.1120717337 0.1921654975 0.2669948024 0.1921654975 0.1120717337 0.0622653676

Table 5

$$w(x) = \begin{cases} 1.5, & 0 \le x < 1/3 \\ 0, & 1/3 \le x < 2/3 \\ 1.5, & 2/3 \le x \le 1 \end{cases}$$

n	x_k	A_k
1	0.5000000000	1.0000000000
2	0.2685930991 0.8183634310	0.5790844061 0.4209155939
3	0.1304685506 0.5475716836 0.8970241350	0.3197058931 0.4348282482 0.2454658587
4	0.0757494353 0.3755358385 0.7037464438 0.9359419648	0.1968595442 0.3578063329 0.2846221764 0.1607119465
5	0.0510480585 0.2606512006 0.5258401241 0.7844367246 0.9557685302	0.2076033849 0.0385479510 0.5503636418 0.0264585191 0.1770265033
6	0.0359392627 0.1818684791 0.4107863017 0.6473261453 0.8397717100 0.9679429371	0.0912099813 0.1962792706 0.2473754400 0.2165465922 0.1668817340 0.0817069819
7	0.0268900596 0.1372789394 0.3275496101 0.5189858847 0.7212279935 0.8769647653 0.9757007050	0.0667733402 0.1577849185 0.2012930105 0.1937620656 0.1914051542 0.1254402232 0.0635412878

Table 6

$$w(x) = \begin{cases} 0.5, & 0 \le x < 1/3 \\ 1, & 1/3 \le x < 2/3 \\ 1.5, & 2/3 \le x \le 1 \end{cases}$$

4 Examples

There will be one example in this section. We will use the above forty-two formulas to find the numerical value of the definite integral of this example.

Example 1

Find the value of the definite integral of $\int_0^1 e^x dx$. The exact value of this integral is $e - 1 \cong 1.7182818285$.

The numerical results of six sequences of orthogonal polynomials are in following table 1 to table 6.

Seq.	Cal. Value	Error
1	1.6094377560	0.1088440724
2	1.6717452055	0.0465366229
3	2.2490495458	0.5307677173
4	1.2839582688	0.4343235596
5	1.7239956217	0.0057137933
6	1.5581046763	0.1601771522

Table 1 One Point

Seq.	Cal. Value	Error
1	17166955350	0.0015862934
2	1.7175241521	0.0007576763
3	1.8495870765	0.1313052480
4	1.5619658430	0.1563159855
5	1.7174373640	0.00084446450
6	1.7170219221	0.00125990640

Table 2 Two Points

Seq.	Cal. Value	Error
1	1.7174486963	0.00083313212
2	1.7178204304	0.00046139811
3	1.9565479956	0.23826616714
4	2.4553418617	0.73706003321
5	1.7182729217	0.00000890676
6	1.7182733379	0.00000849056

Table 3 Three Points

Seq.	Cal. value	Error
1	1.7174687850	0.00081304346
2	1.7178451667	0.00043666177
3	1.7504438839	0.03216205540
4	1.6679239491	0.05035787938
5	1.7182976508	0.00001582234
6	1.7182817265	0.00000010194

Table 4 Four Points

Seq.	Cal. value	Error
1	4.5469097570	2.82862792850
2	1.7177401328	0.00054169566
3	1.7390685189	0.02078669041
4	1.6841009484	0.03418088002
5	1.7182817422	0.00000008621
6	1.7182818367	0.00000000826

Table 5 Five Points

Seq.	Cal. value	Error
1	1.7175256096	0.00056218813
2	1.7178798885	0.00040193997
3	1.8424288684	0.12414704000
4	1.6935541559	0.02472767255
5	1.7182807388	0.00000108964
6	1.7182776358	0.00000419278

Table 6 Six Points

Seq.	Cal. Value	Error
1	1.7174597543	0.0008220742
2	1.7182226383	0.0000591902
3	1.7290520761	0.0107702477
4	1.6995598772	0.0187219512
5	1.7182818284	0.0000000000
6	1.7181937670	0.0000880615

Table 7 Seven Points

5. Conclusion

The results from the example, indicated that the first, second, fifth and sixth sequences of the orthogonal polynomials give the expected results but not the third and forth sequences of the orthogonal polynomials. We strongly recommend the fifth sequence and the sixth sequence.

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