On Linear and Positive Operators

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Abstract: In order to approximate function $f : [0, \infty) \to \mathbb{R}$, with $|f(x)| \le Mx^\alpha$ for $M = M(f) > 0$ and $x > 0$, we introduce the approximation operators $\mathcal{F}_n : f \to \mathcal{F}_n f$, with

$$(\mathcal{F}_n f)(x) = \frac{(nx)_{n+1}}{n!} \int_0^1 t^{nx-1}(1-t)^n f\left(\frac{t}{1-t}\right) dt, \;\; x > 0, \;\; \alpha > 0,$$

where $n \ge n_0$ with $n_0 = [\alpha] + b + 1$ and $n \in \mathbb{N}^*$ - be fixed. Our aim is to find some properties for the above operator.

Key–Words: Approximation Theory, Linear Positive Operators

1 Introduction

Let Y_α linear space of all function

$$f : [0, \infty) \to \mathbb{R},$$

with the property that M exists, $M = M(f) > 0$ and $\alpha > 0$ such that $|f(x)| \le Mx^\alpha$, for all $x > 0$. We define the operators $\mathcal{F}_n : f \to \mathcal{F}_n f$, where $n \ge n_0$, n_0 is noted,

$$(\mathcal{F}_n f)(x) = \frac{(nx)_{n+1}}{n!} \int_0^1 t^{nx-1}(1-t)^n f\left(\frac{t}{1-t}\right) dt, \qquad (1)$$

where $x > 0$, $\alpha > 0$ and $n \ge n_0$ with $n_0 = [\alpha]+b+1$, $n \in \mathbb{N}^*$ - fixed.

2 Main Results

In the following we prove if $f \in Y_\alpha$, then it's image of f that is $\mathcal{F}_n f \in Y_\alpha$.

Theorem 1 *If $(\mathcal{F}_n)_{n \ge n_0}$ are operators defined by (1) and $f \in Y_\alpha$, $|f(x)| \le M(f)x^\alpha$, $\alpha > 0$, $x > 0$ then it exists $M(\mathcal{F}_n f) > 0$ so that $\forall x > c > 0$ we have*

$$|(\mathcal{F}_n f)(x)| \le M(\mathcal{F}_n f)x^\alpha,$$

where $M(\mathcal{F}_n f) = M(f)e^{\frac{2\alpha^2}{b\sqrt{c}}}$

Proof: We have successively

$$|(\mathcal{F}_n f)(x)| \le$$

$$\le \frac{(nx)_{n+1}}{n!} \int_0^1 t^{nx-1}(1-t)^n \left| f\left(\frac{t}{1-t}\right)\right| dt \le$$

$$\le M \frac{(nx)_{n+1}}{n!} \int_0^1 t^{nx-1}(1-t)^n \left(\frac{t}{1-t}\right)^\alpha dt =$$

$$= M \frac{(nx)_{n+1}}{n!} \int_0^1 t^{nx+\alpha-1}(1-t)^{n-\alpha+1-1} dt =$$

$$= M \frac{\Gamma(nx+n+1)}{n!\Gamma(nx)} \frac{\Gamma(nx+\alpha)\Gamma(n-\alpha+1)}{\Gamma(n+nx+1)}.$$

So, we have

$$|(\mathcal{F}_n f)(x)| \le M \frac{\Gamma(n-\alpha+1)\Gamma(nx+\alpha)}{\Gamma(n+1)\Gamma(nx)} \qquad (2)$$

In the following we use the next theorem

Theorem 2 *(Bohr & Mollerup) There is only one function $g : (0, \infty) \to (0, \infty)$ which verifies*

1. $g(1) = 1$
2. $g(x + 1) = xg(x)$
3. $\ln g$ *is convex on* $(0, \infty)$,

then $g(x) = \Gamma(x)$, $\forall\, x > 0$.

From theorem 2 it follows that for all $0 < x_1 < x_2 < x_3 < \infty$ we have

$$[x_1, x_2, x_3; \ln \Gamma] \ge 0, \qquad (3)$$

or

$$(\Gamma(x_2))^{x_3-x_1} \geq (\Gamma(x_1))^{x_3-x_2}(\Gamma(x_3))^{x_2-x_1} \quad (4)$$

We choose $0 < x_1 = z+1-\alpha < x_2 = z+1 < x_3 = z+2 < \infty$. From (4) we have

$$(\Gamma(z+1))^{1+\alpha} \geq (\Gamma(z+1-\alpha))((z+1)\Gamma(z+1))^{\alpha}$$

so

$$\frac{\Gamma(z+1)}{\Gamma(z+1-\alpha)} \leq (z+1)^{\alpha}, \ \ \forall\, z+1 > \alpha > 0. \quad (5)$$

We choose $0 < x_1 = z-\alpha < x_2 = z+1-\alpha < x_3 = z+1-\alpha < z+1 < \infty$, it follows

$$(\Gamma(z+1-\alpha))^{1+\alpha} \geq (\frac{\Gamma(z+1-\alpha)}{z-\alpha})^{\alpha}\Gamma(z+1),$$

so

$$\frac{\Gamma(z+1-\alpha)}{\Gamma(z+1)} \leq \frac{1}{(z-\alpha)^{\alpha}}, \ \ \forall\, z > \alpha > 0. \quad (6)$$

From (5) and (6) we have

$$\frac{1}{(z+1)^{\alpha}} \leq \frac{\Gamma(z+1-\alpha)}{\Gamma(z+1)} \leq \frac{1}{(z-\alpha)^{\alpha}}, \quad (7)$$

for all $z > \alpha > 0$. We choose in (4) $0 < x_1 = nx < x_2 = nx+\alpha < x_3 = nx+\alpha+1 < \infty$, it follows

$$\left(\frac{\Gamma(nx+\alpha+1)}{(nx+\alpha)}\right)^{\alpha+1} \leq (\Gamma(nx))(\Gamma(nx+\alpha+1))^{\alpha},$$

so

$$\frac{\Gamma(nx+\alpha+1)}{\Gamma(nx)} \leq (nx+\alpha)^{\alpha+1}, \ \ \forall\, x > 0, \ \alpha > 0,$$

$$\frac{(nx+1\alpha)\Gamma(nx+\alpha)}{\Gamma(nx)} \leq (nx+\alpha)^{\alpha+1},$$

follows

$$\frac{\Gamma(nx+\alpha)}{\Gamma(nx)} \leq (nx+\alpha)^{\alpha}, \ \ \forall\, x > 0, \ \alpha > 0. \quad (8)$$

We choose in (4) $0 < x_1 = nx-1 < x_2 = nx < x_3 = nx+\alpha < \infty$, it follows

$$\Gamma(nx)^{\alpha+1} \leq \left(\frac{\Gamma(nx)}{nx-1}\right)^{\alpha}\Gamma(nx+\alpha),$$

so

$$0 < (nx-1)^{\alpha} \leq \frac{\Gamma(nx+\alpha)}{\Gamma(nx)}, \quad (9)$$

with $nx-1 > 0$ and $\alpha > 0$. From (8) and (9) we have

$$(nx-1)^{\alpha} \leq \frac{\Gamma(nx+\alpha)}{\Gamma(nx)} \leq (nx+\alpha)^{\alpha}, \quad (10)$$

for all $x > \frac{1}{x}$ and $x > 0$, $\alpha > 0$. So using (7) and (10) in (2) we obtain

$$|(\mathcal{F}_n f)(x)| \leq M(f)\frac{(nx+\alpha)^{\alpha}}{(n-\alpha)^{\alpha}} =$$

$$= M(f)x^{\alpha}\frac{(nx+\frac{\alpha}{x})^{\alpha}}{(n-\alpha)^{\alpha}} = M(f)x^{\alpha}\left(1+\frac{\alpha+\frac{\alpha}{x}}{n-\alpha}\right)^{\alpha} =$$

$$= M(f)x^{\alpha}\left(1+\frac{\alpha+\frac{\alpha}{x}}{n-\alpha}\right)^{\frac{n-\alpha}{\alpha+\frac{\alpha}{x}}\frac{\alpha+\frac{\alpha}{x}}{n-\alpha}\alpha},$$

so

$$|(\mathcal{F}_n f)(x)| < M(f)x^{\alpha}e^{\frac{\alpha^2\left(1+\frac{1}{x}\right)}{n-\alpha}} < \quad (11)$$
$$< M(f)x^{\alpha}e^{\frac{2\alpha^2}{\sqrt{x}(n-\alpha)}}.$$

We fix $b \in \mathbb{N}^*$ and define

$$n_0 = [\alpha+b+1] = [\alpha]+b+1 \geq \alpha+b+1.$$

Next $n \geq n_0$ it follows $n-\alpha \geq b+1$, that is

$$\frac{1}{n-\alpha} \leq \frac{1}{b+1} < \frac{1}{b}.$$

From (11) it follows

$$|(\mathcal{F}_n f)(x)| \leq M(f)x^{\alpha}e^{\frac{2\alpha^2}{\sqrt{x}b}} \leq$$

$$\leq M(f)x^{\alpha}e^{\frac{2\alpha^2}{b\sqrt{c}}} =: M(\mathcal{F}_n f)x^{\alpha},$$

with $M(\mathcal{F}_n f) = M(f)x^{\alpha}e^{\frac{2\alpha^2}{b\sqrt{c}}}$.

Next we calculate $(\mathcal{F}_n e_j)(x)$, where $e_j(x) = x^j$. We obtain successively

$$(\mathcal{F}_n e_j)(x) = \frac{(nx)_{n+1}}{n!}\int_0^1 t^{nx+j-1}(1-t)^n dt =$$

$$= \frac{\Gamma(nx+n+1)}{\Gamma(nx)n!}B(nx+j, n-j+1) =$$

$$= \frac{\Gamma(nx+n+1)}{\Gamma(nx)n!}\frac{\Gamma(nx+j)\Gamma(n-j+1)}{\Gamma(nx+n+1)} =$$

$$= \frac{(nx)_j\Gamma(n-j+1)}{\Gamma(n+1)} = \frac{(nx)_j}{(n-j+1)_j}.$$

So we have $(\mathcal{F}_n e_0)(x) = 1$, $(\mathcal{F}_n e_1)(x) = x$, respectively

$$(\mathcal{F}_n e_2)(x) = \frac{(nx)(nx+1)}{(n-2+1)_2} = \frac{nx(nx+1)}{n(n-1)} =$$

$$= x^2 + \frac{x(1-x)}{n-1} \stackrel{n\to\infty}{\longrightarrow} x^2$$

Starting from the theorem

Theorem 3 *(A. Lupaş [3]) If*

$$\lim_{n\to\infty}(\mathcal{L}e_j)(x) = [\varphi(x)]^j, \ \ j = 0, 1, 2$$

then

$$\lim_{n\to\infty}(\mathcal{L}f)(x) = f(\varphi(x)),$$

for f continuous on $[0, M]$, $M > 0$.

Using theorem 3 it follows the following statement is true

Theorem 4 *Let $f : [0, \infty) \to \mathbb{R}$, with*

$$|f(x)| \leq Mx^{\alpha}, \ \ \alpha > 0, \ \ M > 0,$$

when $x \to \infty$. If $\mathcal{F}_n$ are positive linear operators defined as is (1), then

$$\lim_{n\to\infty}(\mathcal{F}f)(x) = f(x),$$

for f continuous on $[0, M]$, $M > 0$.

A. Lupaş [5] prove the following proposition

Theorem 5 *If*

$$L : C(K) \to C(K_1), \ \ K_1 = [a_1, b_1] \subseteq K,$$

is a positive linear operator, then for all $f \in C(K)$ and $\delta > 0$ we have

$$||f - Lf||_{K_1} \leq ||f|| \cdot ||e_0 - Le_0||_{K_1} +$$

$$+ \inf_{m=1,2,\ldots}\{||Le_0||_{K_1} + \delta^{-m}||L\Omega_m||_{K_1}\}\omega(f;\delta),$$

where $||\cdot|| = \max\limits_{K}|\cdot|$ and $\Omega_m(t) = (t - x)^m$.

We have the following theorem

Theorem 6 *If $\mathcal{F}_n f$ is define by (1), then for all $f \in Y_\alpha \cap C[0, \infty)$, $\alpha \geq 2$ we have*

$$||f - \mathcal{F}_n f|| \leq \frac{5}{4}\omega\left(f; \frac{1}{\sqrt{n-1}}\right)$$

Proof: We consider $m = 2$, $\Omega_2(t) = (t - x)^2$.

Taking into consideration the above we have (see [6], [8])

$$(\mathcal{F}_n\Omega_2)(x) = x^2 + \frac{x(1-x)}{n-1} - 2x^2 + x^2 = \frac{x(1-x)}{n-1},$$

choosing $\delta = \frac{1}{\sqrt{n-1}}$ and using the fact that

$$x(1-x) \leq \frac{1}{4},$$

it follows

$$||f - \mathcal{F}_n f|| \leq \frac{5}{4}\omega\left(f; \frac{1}{\sqrt{n-1}}\right).$$

In the following we consider the Post-Widder positive operator

$$(W_n f)(x) = \frac{1}{\Gamma(nx)}\int_0^\infty e^{-t}t^{nx-1}f\left(\frac{t}{n}\right)dt, \quad (12)$$

Theorem 7 *For $\mathcal{F}_n f$ operators defined in (1) we have*

$$\mathcal{F}_n f = W_n G_n f,$$

where W_n are Post Widder operators, $G_n f$ Gamma operators respectively.

Proof: Starting from representation of Gamma operators

$$(G_n f)(x) = \frac{1}{n!}\int_0^\infty e^{-t}t^n f\left(\frac{nx}{t}\right)dt,$$

for $x = \frac{t}{n}$ we have

$$(G_n f)\left(\frac{t}{n}\right) = \frac{1}{n!}\int_0^\infty e^{-s}s^n f\left(\frac{t}{s}\right)ds.$$

We calculate successively

$$(W_n G_n f)(x) = \frac{1}{n!\Gamma(nx)}\cdot$$

$$\cdot\int_0^\infty e^{-t}t^{nx-1}\left(\int_0^\infty e^{-s}s^n f\left(\frac{t}{s}\right)ds\right)dt =$$

$$= \frac{1}{n!\Gamma(nx)}\int_0^\infty e^{-s}s^n\cdot$$

$$\cdot\left(\int_0^\infty e^{-t}t^{nx-1}f\left(\frac{t}{s}\right)dt\right)ds.$$

we chance the variable $\frac{t}{s} = y$, that is $t = ys$ and we have

$$(W_n G_n f)(x) = \frac{1}{n!\Gamma(nx)}\int_0^\infty e^{-s}s^{n+nx}\cdot$$

$$\cdot\left(\int_0^\infty e^{-ys}y^{nx-1}f(y)dy\right)ds = \frac{1}{n!\Gamma(nx)}\cdot$$

$$\cdot\int_0^\infty\left(\int_0^\infty e^{-s(1+y)}s^{n+nx}ds\right)y^{nx-1}f(y)dy,$$

we note that $s(1+y) = T$, $ds = \dfrac{1}{1+y}dT$ and we have

$$(W_n G_n f)(x) = \frac{1}{n!\Gamma(nx)}\int_0^\infty \frac{1}{(1+y)^{n+nx+1}}\cdot$$

$$\cdot \left(\int_0^\infty e^{-T} T^{n+nx} dT \right) y^{nx-1} f(y) dy,$$

but

$$\Gamma(n+nx+1) = \int_0^\infty e^{-T} T^{n+nx} dT,$$

it follows

$$(W_n G_n f)(x) =$$

$$= \frac{\Gamma(n+nx+1)}{n!\Gamma(nx)} \int_0^\infty \frac{y^{nx-1}}{(1+y)^{n+nx+1}} f(y) dy =$$

$$= \frac{(nx)_{n+1}}{n!} \int_0^\infty \frac{y^{nx-1}}{(1+y)^{n+nx+1}} f(y) dy.$$

We note $\frac{y}{1+y} = t$, $dy = \frac{1}{(1-t)^2} dt$, we have

$$(W_n G_n f)(x) = \frac{(nx)_{n+1}}{n!} \cdot$$

$$\cdot \int_0^1 \left(\frac{t}{1-t} \right)^{nx-1} \frac{(1-t)^{n+nx+1}}{(1-t)^2} f\left(\frac{t}{1-t} \right) dt =$$

$$= \frac{(nx)_{n+1}}{n!} \int_0^1 t^{nx-1} (1-t)^n f\left(\frac{t}{1-t} \right) dt =$$

$$= (\mathcal{F}_n f)(x).$$

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