Interaction of Discrete Oscillatory Agents and Determination of Stability Regions by Lyapunov Exponents

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Abstract: Interaction of discrete dynamics Kaldor's anticipative cobweb agents is analyzed. Interaction rule between agents is determined by the floor-roof principle. As the main property the synchronization emerges. Emergent synchronization rule was determined by application of *z-transform*. Nonlinear expansion of anticipative model provided emergent attractor response. Determination of stability is explored by the regions where Lyapunov exponents are negative.

Key–Words: Lyapunov exponents, nonlinear system, anticipative, discrete, Kaldor, Cobweb, agent

1 Introduction

Anticipative formulation of the Kaldor's cobweb system is developed according to the hyperincursivity paradigm [1] which is applied for the explanation of the emergent holonomic properties of natural and artificial systems. A particular case of hyperincursivity is the classical well-known concept of recursive loop.

In order to define the *incursivity* let us first describe the general formulation of recursive discrete system, which computes its successive time states as a function of its past and present states as:

$$x(t+1) = \mathbf{R}(\ldots, \mathbf{x}(t-2), \mathbf{x}(t-1), \mathbf{x}(t); \mathbf{p}) \quad (1)$$

where $\mathbf{x}(t)$ are the vector states at time t, $\mathbf{R}$ the recursive vector function and $\mathbf{p}$ a set of parameters. In knowing the function $\mathbf{R}$, the values of parameters $\mathbf{p}$ and the initial conditions $\mathbf{x}(-2), \mathbf{x}(-1), \mathbf{x}(0)$ the successive states $x(1), x(2), x(3), \ldots$ can be recursively computed where the interval of time $\Delta t = 1$ is a duration.

A recursive differential system is a system such as

$$\frac{d\mathbf{x}}{dt} = \mathbf{R}(\mathbf{x}(t); \mathbf{p}) \quad (2)$$

The numerical computation of such differential equations are based on discrete equations obtained from the definition, at one hand, of the forward derivative as

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \Delta t \mathbf{R}(\mathbf{x}(t); \mathbf{p}) \quad (3)$$

which is a discrete recursive system and at the other hand, from the backward derivative as

$$\mathbf{x}(t) = \mathbf{x}(t-\Delta t) + \Delta t \mathbf{R}(\mathbf{x}(t); \mathbf{p}) \quad (4)$$

where Δt is a time interval. This backward derivative is the forward derivative in replacing Δt by $-\Delta t$ in the following way: $\mathbf{x}(t) = \mathbf{x}(t+\Delta t) - \Delta t \mathbf{R}(\mathbf{x}(t); \mathbf{p})$, which can be with $t = t + \Delta t$ rewritten as

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \Delta t \mathbf{R}(\mathbf{x}(t+\Delta t); \mathbf{p}) \quad (5)$$

where the state at time $t + \Delta t$ depends on the state at the present time t but also at the future time $t + \Delta t$. This represents a self-referential anticipatory system which is by D. Dubois [1] characterized as an incursive system from the contraction of "inclusive" or "implicit" recursion [2].

Definition 1 *Incursive system is defined by:*

$$x(t+1) = F[\ldots, x(t-1), x(t), x(t+1), \ldots] \quad (6)$$

where the value of a variable $x(t+1)$ at time $t+1$ is a function of this variable at past, present and future times.

Incursivity paradigm should be considered as the idea for the examination of the system structure and its relation to the time component. Application of incursivity Def. 1 to the classical Kaldor's cobweb model yields the following set of equations for the anticipative cobweb model:

$$P(k+2) = \frac{d}{b}\Big(P(k+1) - \big(\frac{b\,P(k) - c + a}{d}\big)\Big) \quad (7)$$

$$Q_s(k+2) = \frac{d}{b}\Big(Q_s(k+1) - \big(a + \frac{b}{d}(Q_s(k) - c)\big)\Big) \quad (8)$$

What does actually the described modification represent? This is actually the introduction of control mechanism that is based on the difference of the system states rather than the states itself which is in fact the *feedback-anticipative* control mechanism. If we reformulate Eq. 7 and Eq. (8) the dependency of the future-present-past events could be observed:

$$P(k) = \frac{bP(k-1) + a - c}{d} + \frac{b}{d}P(k+1) \quad (9)$$

$$Q_s(k) = \frac{b}{d}Q_s(k+1) + \frac{b}{d}Q_s(k-1) + a - \frac{bc}{d} \quad (10)$$

Eq. (9) and Eq. (10) state that the value of the present is dependent on the past as well as on the future. This paradoxical statement is realizable since the formulation of feedback anticipative chain could be stated. Considered system of equations has two delay chains, one for P and one for Q_s. The model stated in the *incursive* form considers, that the state value depends not only on the state value in the time $k-1$ but also on the state value in time $k+1$. Therefore the change in supply and demand should be dependent not only on the values in the past, the future should also be considered at the determination of present supply and demand values. Abstracted procedure in the above case is revealed by considering classical Kaldor's cobweb model and Eqs. (7) and (8). Here the equation for the difference operator Δ has been transformed to the state equation while the time arguments that were applied are in the minimal incursive form $\{t-1, t, t+1\}$; here t represents discrete time k.

2 Agent Based System

Economic systems are determined by their periodic response. Let us suppose that there are many different economic systems in our environment which interact. Here the economic system will be represented as an agent A interacting with other agents with goal to reach equilibrium by differentiating parameter d which determines frequency response of the system. Here the question arises, is it possible to control the economic system by changing the control parameter d which actually alters the frequency response of the system. The deviation in prices could be understood as the change of the frequency response of the system. In other words, if we change the price, the frequency of the system alters. Therefore one should try to control the system by changing it's frequency response.

Consider the following agent based anticipative cobweb model of price P dynamics derived from equations Eq. (9):

$$\mathbf{P}(k) = \frac{b\,\mathbf{P}(k-1) + a - c}{\mathbf{d}(k)} + \frac{b}{\mathbf{d}(k)}\mathbf{P}(k+1) \quad (11)$$

Initial conditions for Eq. (11) should be stated in matrix form. In above equations matrix annotation represents column vectors which have the same arbitrary dimension n determined by the number of agents A_n. The decision of change in parameter d will be dependant on the sum of two price values at time $k+1$ and time $k-1$. Here the relative value of the price by taking the range of systems response in the nominator will be considered:

$$\mathbf{e} = \frac{\mathbf{P}(k+1) + \mathbf{P}(k-1)}{||\lceil \mathbf{P(k)} \rceil|| + ||\lfloor \mathbf{P(k)} \rfloor||} \quad (12)$$

In order to perform the control by variation of parameter d where n agents are present the following state equation with the adaptive rule for $\Delta\mathbf{d}(k)$ is introduced:

$$\mathbf{d}(k+1) = \mathbf{d}(k) + \Delta\mathbf{d}\,(k) \quad (13)$$

where $\Delta\mathbf{d}$ determines the change in control parameter d:

$$\Delta\mathbf{d}(k) = \begin{cases} \beta & \text{if } \; e = \lceil \mathbf{e} \rceil \\ -\beta & \text{if } \; e = \lfloor \mathbf{e} \rfloor \end{cases} \quad (14)$$

In above definition of agent's rule the *floor* and *ceiling* functions over a vector of relative prices $\mathbf{e}$ considers only a finite number of lags. Parameter β is the *intensity* of agents' reaction to the market disequilibrium; $\beta \in (0,1)$. Initialization of vector $\mathbf{d}$ is determined by random value $r_i \in [-2, 2]$ which is in the interval of periodic solutions for the anticipative cobweb system. Idea captured in above definition considers situation where economic system of which the market price in past and estimated future is the highest should be controlled by increasing the value of control parameter d changing the frequency response of the system. The case on the lower end of the market price is inverse.
The system stability is dependent on the value of agents reaction $\beta \in (0,1)$. Higher values of β result in higher volatility of systems response. Another important variable is the number of economic agents

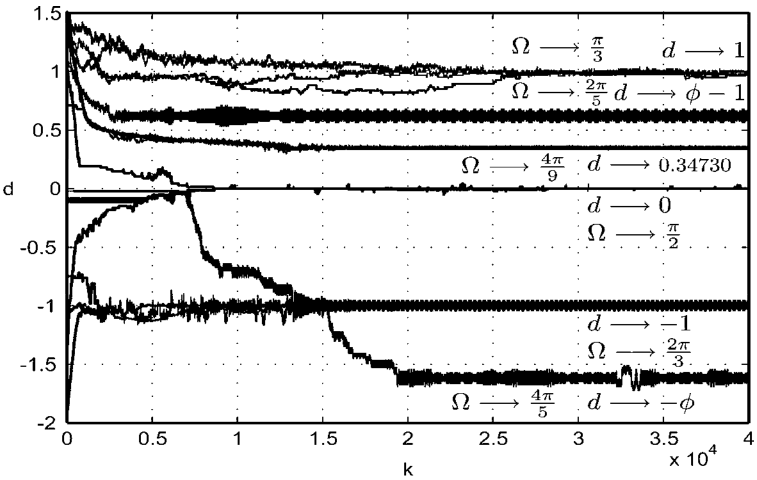

Figure 1: Convergence of control parameter d values of agent based anticipative cobweb model

n considered. Higher number of agents contribute to the modest changes in parameter d value.

Fig.1 shows the examples of system response at the synchronization. In the considered model up to 40 agents interacted and the response of the control parameter d which is shown on $y - axis$ is represented *vs* discrete time step k. One could observe the partial limits to the values for example $d = 1$, $d = \phi - 1$, $d = 0.34730$, $d = 0$, $d = -1$ and $d = -\phi$. There is a question, what actually determines the synchronization values of the interacting agents?

The observed system has some sort of *'gravitational'* tendency i.e. the values of parameter d are attracted to one another and mostly trapped in the region $d \in [-2, 2]$ where the response of the system is periodic. The significant plateaus in response occur which could be observed in the real economic systems which stabilize and consequently step above or below the equilibrium value.

3 Origins of Synchronization

The z-transform is the basis of an effective method for solution of linear constant-coefficient difference equations. It essentially automates the process of determining the coefficients of the various geometric sequences that comprise a solution [3]. The application of z-transform on the Eq.7 and Eq.8 with proper initial conditions gives:

$$Y(z) = \frac{-y_1 z + y_0 d z - y_0 z^2}{-1 + dz - z^2} \quad (15)$$

Inverse z-transform yields the following solution:

$$\begin{aligned} Y^{-1}(z) = {} & 2^{-1-n} y_0 \left(d - \sqrt{-4+d^2}\right)^n - \\ & - \frac{y_1 \left(d - \sqrt{-4+d^2}\right)^n}{2^n \sqrt{-4+d^2}} + \\ & + \frac{2^{-1-n} y_0 d \left(d - \sqrt{-4+d^2}\right)^n}{\sqrt{-4+d^2}} + \\ & + 2^{-1-n} y_0 \left(d + \sqrt{-4+d^2}\right)^n + \\ & + \frac{y_1 \left(d + \sqrt{-4+d^2}\right)^n}{2^n \sqrt{-4+d^2}} - \\ & - \frac{2^{-1-n} y_0 d \left(d + \sqrt{-4+d^2}\right)^n}{\sqrt{-4+d^2}} \end{aligned} \quad (16)$$

In order to gain conditions for the periodic response of the system the following equation should be solved:

$$Y^{-1}(z) = y_0 \quad (17)$$

Let us compute a numerical example of periodic solution applying the *z-transform*. The period examined will be the period of 9 i.e. $n = 9$. In Eq.17 one should put the condition $n = 9$. One of the possible solutions for the initial condition worth of examination is the following:

$$d = \frac{1}{\left(\frac{1}{2}(-1 + i\sqrt{3})\right)^{\frac{1}{3}}} + \left(\frac{1}{2}(-1 + i\sqrt{3})\right)^{\frac{1}{3}} \quad (18)$$

The term $(-1 + i\sqrt{3})^{\frac{1}{3}}$ (let us denote the term as z^*) could be expressed in the following way by three different imaginary values in polar form:

$$z_1^* = \sqrt[3]{2}\Big(\cos\frac{2\pi}{9} + i \sin\frac{2\pi}{9}\Big) \quad (19)$$

$$z_2^* = \sqrt[3]{2}\Big(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\Big) \quad (20)$$

$$z_3^* = \sqrt[3]{2}\Big(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\Big) \quad (21)$$

By putting Eq.19, Eq.20 and Eq.21 into Eq.18 and performing trigonometric reduction one gets the following solutions:

$$d_1 = 2\cos\frac{2\pi}{9} \quad d_2 = 2\cos\frac{4\pi}{9} \quad d_3 = 2\cos\frac{8\pi}{9} \quad (22)$$

By inspecting the Eq.18 and considering the equation for the roots of complex numbers [4]:

$$\sqrt[n]{z} = \sqrt[n]{r}\Big(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\Big) \quad (23)$$

the general form of the solution for the parameter d could be assumed:

$$d = 2\cos\frac{2\pi m}{n} \quad (24)$$

where n is the period and $m = 1, 2, 3, ..., n-1$. Similar procedure could be performed for the arbitrary period n. More general solutions which regards also the parameter b which was fixed for the purpose of determination of solutions is:

$$d = 2b\cos\frac{2\pi m}{n} \quad (25)$$

Numerical values of the solutions for parameter d confirm the findings of Sonis [6, 5] characterized by Def. 2 about the domain of attraction for 2D dynamics by n-dimensional linear bifurcation analysis. Determination of the periodicity condition for the anticipative system of cobweb linear difference equations leads to the following proposition which provides the interconnection between the periodicity cycles in considered anticipative cobweb model and general bifurcation conditions for the nonlinear discrete dynamical systems:

Definition 2 *Periodicity conditions of the anticipative cobweb model* $d = 2b\cos\frac{2\pi m}{n}$ *equals bifurcation condition on the flutter boundary determining q-periodic fixed points of the nonlinear discrete dynamical system* $trJ = 2\cos 2\pi\Omega$.

Here Ω represents a rational fraction $\Omega = \frac{p}{q}; b = 1$. Def. 2 provides dynamical interpretation of bifurcation periodicity conditions which are important at the analysis of non-linear systems of higher complexities where manifestations of chaos and turbulence might occur [6]. There is a significant effort present to further determination of conceptual framework of bifurcation analysis for which the central problem is the bifurcation control. Def. 2 provides possible simplification of bifurcation conditions representations. Geometrical visualization of such conditions is important factor at the analysis of the nonlinear discrete dynamical systems [7].

In order to explain the bifurcation condition in Def. 2 general notation for two-dimensional discrete map will be applied:

$$\begin{cases} P_{1_{k+1}} = f(P_{1_k}, P_{0_k}) \\ P_{0_{k+1}} = g(P_{1_k}, P_{0_k}) \end{cases} \quad (26)$$

where P_{1_k} and P_{0_k} represent the components of the iteration process at time k. Let us assume, that the equilibrium values of the system are: P_1^* and P_0^*. In general case the linear approximation near the equilibrium point should be considered to determine the characteristics of the system:

$$\begin{pmatrix} \Delta P_{1_{k+1}} \\ \Delta P_{0_{k+1}} \end{pmatrix} = \begin{pmatrix} \Delta P_{1_k} \\ \Delta P_{0_k} \end{pmatrix} \quad (27)$$

where $\Delta P_{0_k} = P_{0_k} - P_0^*$ and $\Delta P_{1_k} = P_{1_k} - P_1^*$ denote differences of variables at time k from the equilibrium point, and J is the the Jacobian matrix which should be evaluated at assumed equilibrium points:

$$J = \begin{pmatrix} \frac{\partial f(P_1^*,P_0^*)}{\partial P_1} & \frac{\partial f(P_1^*,P_0^*)}{\partial P_0} \\ \frac{\partial g(P_1^*,P_0^*)}{\partial P_1} & \frac{\partial g(P_1^*,P_0^*)}{\partial P_0} \end{pmatrix} \quad (28)$$

Stability result corresponds to the polynomial characteristic equation for which the periodic solutions will be considered,

$$\lambda^2 = \mathrm{tr}J\lambda - \det J \quad (29)$$

where,

$$\mathrm{tr}J = \frac{\partial f(P_1^*, P_0^*)}{\partial P_1} + \frac{\partial g(P_1^*, P_0^*)}{\partial P_0} \quad (30)$$

and

$$detJ = \frac{\partial f(P_1^*, P_0^*)}{\partial P_1}\frac{\partial g(P_1^*, P_0^*)}{\partial P_0} \quad (31)$$

$$- \frac{\partial g(P_1^*, P_0^*)}{\partial P_1}\frac{\partial f(P_1^*, P_0^*)}{\partial P_0} \quad (32)$$

Stability of the system is characterized by the following condition:

$$|\lambda_1| < 1 \text{ and } |\lambda_2| < 1 \quad (33)$$

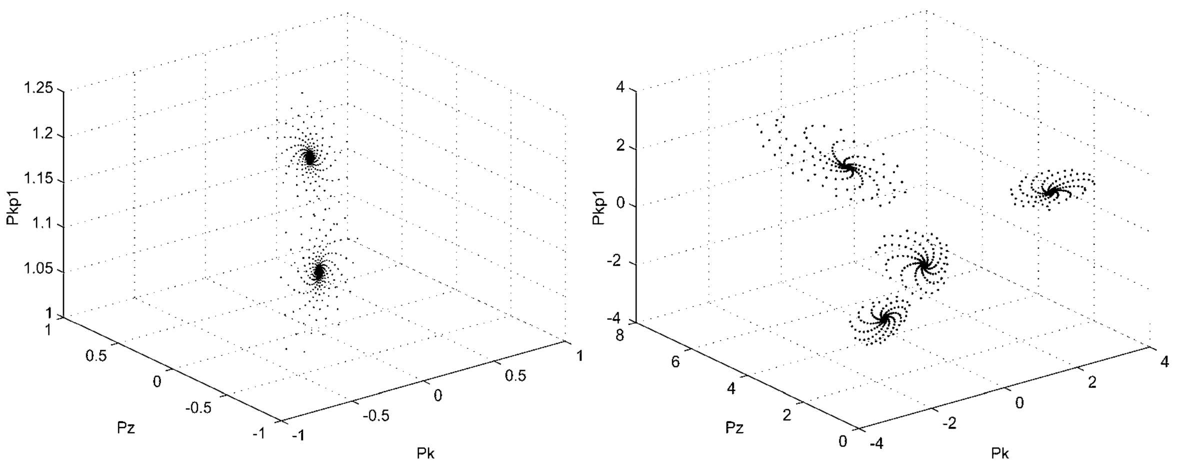

Figure 2: *left)* Emergence of two Synchronous Attractors in the nonlinear case where $d = 0.3833$ and $b_1 = 0.33$ and *right)* Emergence of Four Synchronous Attractors where $d = 0.160793$ and $b_1 = 0.18$

According to the Routh-Hurwitz algorithm we have the following stability boundaries:
a) divergence boundary:

$$\mathrm{tr}J = \det J + 1 \tag{34}$$

b) flip boundary:

$$-\mathrm{tr}J = \det J + 1 \tag{35}$$

c) flutter boundary:

$$\det J = 1 \tag{36}$$

Here the periodicity condition could be stated as $trJ = 2\cos 2\pi\Omega$ which is applied in Def. 2. One should consider e.g. [5] for details.

The range of the cyclical behavior is determined by the classical imaginary solution of the dynamical system which is in our case defined by the characteristic equation

$$\lambda = \frac{-2b + d \mp \sqrt{-4b^2 + d^2}}{2b} \tag{37}$$

According to the terms for λ_1 and λ_2 the following relation is determined: $q = -p$. If we want to determine at what combination of parameters b and d the solution will be periodical i.e. inside the characteristic parabola, the equation $p = \frac{p^2}{4}$ should be considered yielding the definition interval for the value of parameter $d \in [-2b, 2b]$.

Visual representation, frequency response and bifurcation analysis is important at the analysis of complex nonlinear dynamical systems [7]. Frequency mappings of the system and visualization of the periodic solution is therefore important for the analyzing of periodic or chaotic solutions of differential and discrete difference equations. Numerical values of the solutions for parameter d are important since this values confirm the findings of Sonis [6, 5] about the domain of attraction for 2D dynamics by n-dimensional linear bifurcation analysis. One of the important conditions gained by the proposed inspection is the value of the period $n = 10$ which is in close relation to the period $n = 5$. The value of parameter d is $d = \frac{1}{2}(1+\sqrt{5})$ with numerical value 1.61803... This solution represents the "*Golden Ratio*" (ϕ). Some of the different representations of solution for parameter d value at period $n = 10$ are:

$$d_{10} = \phi = 2\cos\frac{\pi}{5} = \frac{1}{2}(1+\sqrt{5}) = 1.618033... \tag{38}$$

The first solution of parameter d at period $n = 10$ connects the considered discrete system with the Fibonacci numbers given by the infinite series:

$$d_{10} = \phi = 1 + \sum_{u=1}^{\infty} \frac{(-1)^{u+1}}{F_u F_{u+1}} \tag{39}$$

The fact, that the periodicity conditions of the examined discrete system incorporates the golden ratio number ϕ could be observed in other studies [8] of complex nonlinear expansions of the basic cob-web systems e.g. Brock and Hommes "Almost Homoclinic Tangency Lemma". One should expect that the symmetric response in $n - mapping$ should follow the pattern with the match in certain point of solution with the ϕ condition. The source of the mentioned condition is presented by the preceding procedure. (The value of parameter d for mentioned period $n = 5$ is $d = \frac{\sqrt{5}-1}{2} = 0.61803...$ often called the "*Golden Mean*".) At the solutions, one should consider, that since the period 2 is on the boundary of the solution the periodic response of the system depends on the initial conditions. Example of numerical values of the period-2 response: $a = 1, b = 1, c = -1, d = -2, p = 1$. The value for tetragon synchronization, which is one of the strongest in the system, is taken in the limit since the system of equation returns the undefined value when $d = 0$ therefore one should consider the tetragon period condition as the value approaching to zero i.e. $d \rightarrow 0$. In this case the system response is undetermined $(\frac{0}{0})$ in it's critical point.

The passage to the understanding of the evolutionary principle in proposed complex agent based model could be obtained through the analysis of systems' frequency response. In our case, by Def. 2 the notion of periodicity transforms to the system's bifurcation analysis. Certainly, the classical Kaldor's cobweb model could not produce periods other than period "2". As it will be shown, the anticipative formulation extended to the nonlinear case is much richer in this regard as it will be presented in the following section.

4 System Modification

If we consider the following proposed socio-economic model by introducing the nonlinearity there are several implications [9, 10, 16]: *a)* neccessity of the nonlinear equation application in order to simulate the system, *b)* evolutionary character of the socio-economic systems could only be revealed via computer simulation and *c)* nonexistence of analytical solutions. The obvious approach to model complex dynamical systems is therefore the mixture of several approaches, continuous simulation, discrete-event simulation and application of nonlinear dynamics models. Important fact that should be considered at the modeling of complex systems is the impossibility of prediction [9] as the immanent characteristic which is present in nonlinear chaotic model representations. Consider generic alteration of the initial anticipative cobweb model:

$$\begin{aligned} P_K(k+1) &= P_K + P_{KP1}(k) - \\ &- \left(P_K(k) + \frac{1}{P_Z(k)P_K(k)}\right) \\ P_{KP1}(k+1) &= P_{KP2}(k) \\ P_{KP2}(k+1) &= \frac{d}{b}\left(P_{KP1} - \frac{bP_K(k) - c + a}{d}\right) \\ P_Z(k+1) &= P_Z(k) + P_K(k)P_{KP1}(k) - \\ &- vP_Z(k) \end{aligned} \quad (40)$$

Slight modification of initial Hicks' model [14, 15] with applied accelerator principle [11] gives the interesting response. The system can be represented in three dimensions which reveals the periodicity of the system for which the previously determined conditions of Farey tree generally still holds. The underlying Farey sequence define the adapted nonlinear *2-d* discrete map. Such evidences are also found in other works in nonlinear system analysis for example [8] or in the recent works of Swedish economist T. Puu.

If we consider the proposed nonlinear system by the examination of Lyapunov exponents defined by Eq. 41 the chaotic property is indicated [17].

$$\frac{1}{n}\left(\ln|f'(x_0)| + \ln|f'(x_1)| + \ldots + \ln|f'(x_{n-1})|\right) \quad (41)$$

Eq. 41 tells us that the Lypunov exponent is the rate of divergence of the two trajectories. Stated differently, we have here the average of the natural logarithm of the absolute value of the derivatives of the map function evaluated at the trajectory points. If the application of the map function to two nearby points leads to two points further apart then the absolute value of the derivative of the map function is greater than 1 when evaluated at those trajectory points. If the absolute value is greater than 1 then the logarithm is *positive*. If the trajectory points continue to diverge on the average, then the average of the logarithm of the derivatives absolute values is positive. In our case the *Gram-Schmidt* orthogonalization procedure [18, 12] which gives the following results for the three Lyapunov exponents, λ_1, λ_2 and λ_3 shown in Fig.3.

From the Fig.3 the stable regions where all three the Lyapunov exponents λ_1, λ_2 and λ_3 are less than 0 could be observed. For the studied system the regions of stability are important. As the example of the application of Lyapunov exponents one should consider parameter bifurcation in the parameter space of interest in order to identify regions of stability. As an example, the computation of the Lyapunov exponents for parameters $d \in [0.1, 4]$ and $v \in [0.5, 0.6]$ was performed indicating the region of stability shown in the Fig. 4. The blue region shows the chaotic parameter space of the system while the white region indicates the stable region. Such stability regions are usually periodic regions obeying previously mentioned Farey sequence rule. At the analysis of nonlinear system the regions of stability are of particular interest [13, 19]. This regions are dependant on the parameter values and initial conditions; here we have the function $f(A, X_0, Y_0, Z_0)$ where A represents set of parameter values, X_0, Y_0 and Z_0 represents the set of initial values. Analytical determination of boundary conditions of periodicity regions is difficult task and in many cases impossible. For the determination of the stability regions, the parameter space

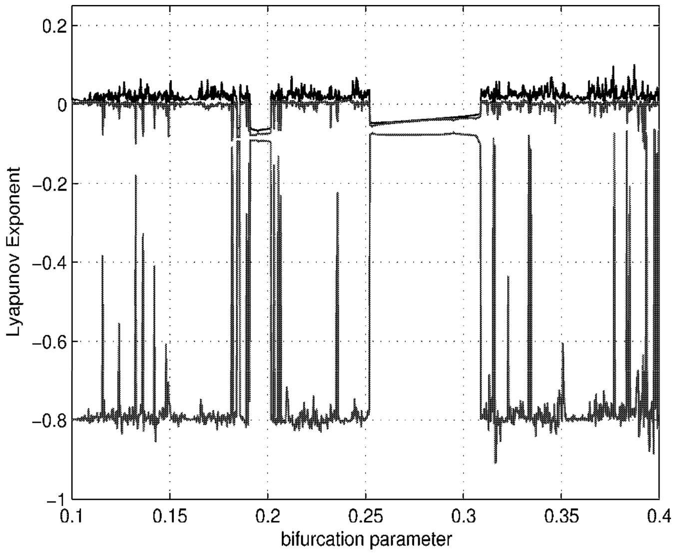

Figure 3: Lyapunov exponents for 3-d discrete nonlinear map for λ_1, λ_2 and λ_3

search is performed as the aggregate Lyapunov estimator dependant on the system response for the function $f(A, X_0, Y_0, Z_0)$. However, one should be aware, that a full understanding of a nonlinear system might require computing solutions for essentially all initial conditions. Since this is infeasible, we would like some simple ways to summarize the possible behaviors as proposed by the Farey tree sequence. Here the application of powerful parallel computing provides the basic platform for systematic [20, 21] nonlinear system analysis. But one should be warned that only the analytical explanations of the nonlinear systems' behavior provide methodologically correct characterization.

5 Conclusion

The Agent based hyperincursive cobweb model enables us to change the future as well as the past chain of events. Emergent synchronization patterns determined by the application of *z-transform* provide the base for determination of stability regions in systems of higher complexities. One of the important results gained is that the proposed agent-based system is apparently controllable by considering the frequency response of the system. Here the main change that is observed in agent-based system is frequency. In our case perturbation was performed by variation of key parameter d. Lyapunov exponents provide the mean for the determination of the stability regions of the system. While observing the single Lyapunov exponent time series the determination of the exponents

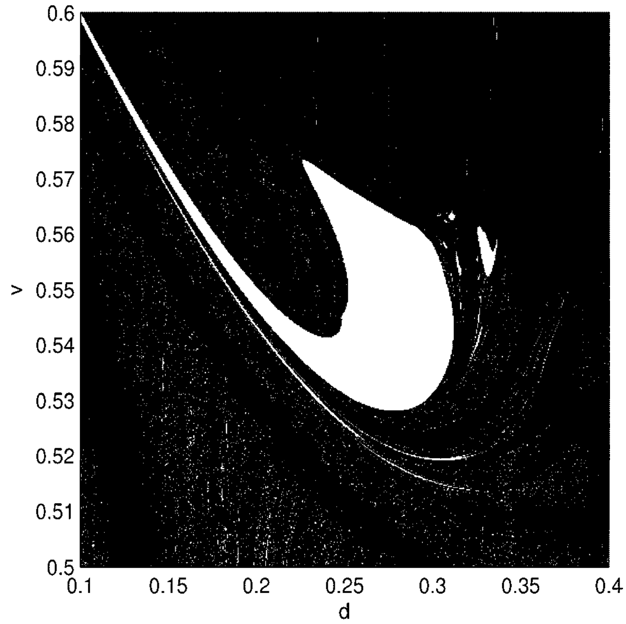

Figure 4: Lyapunov exponents stability region (white) for 3-d discrete nonlinear map for parameter space $d \in [0.1, 4]$ and $v \in [0.5, 0.6]$

in parameter space enable us to determine the general stability regions. Such computations are time demanding and need high processing power. Here the new technologies such as *Grid* and parallel computing should take place in further examination of such systems. One could possibly argue that discrete dynamics is not proper for considering the real world dynamics. On the contrary, many of the systems, that surround us obey the laws of discrete systems. The following procedure proposition emerges which enabled the anticipative formulation of the classical dynamic system: since the hyperincursive systems are hard to determine [22, 1], the developed anticipatory mechanisms should be applied, therefore the model should be a) transformed in the separated form b) provide past-future chain property and c) apply the hyperincursive structure to the studied model. The solution of the periodicity conditions for the *2-d* discrete linear cobweb map provided the means to determine the periodicity conditions for the more complex system. Analytical approach with *z-transformation* provides the proper way to determine the periodic solutions. Negative Lyapunov exponents regions provided the determination of the stability in parameter space.

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