Chaotic behaviour of system on the example of Verhulst model of population

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Abstract: In the contribution the chaotic behaviour of the simple model of dynamic system is discussed. Is shown that the accuracy of numerical solving play significant role in understanding of the chaotic behaviour of the model of dynamic system. This consideration was based on the simple population model described by Verhulst.

Key-Words: simulation, model, dynamic system, chaos, numerical methods

1 Introduction

The first investigator who found chaotic behaviour of the dynamic systems through computer exploration of simple nonlinear system was a meteorologist, Edward Lorentz. According to Gleick's popularization *Chaos,* [4], the first physicist who used computer to investigate a then dimly perceived world of nonlinear dynamics was Mitchel Feingenbaum in the mid-1970s. From that time till todays the chaotic behaviour of the systems is studied in many books, articles and notes for example [1], [2], [4], [5], [6], [7], [9], [10], [11], [12], [13], [14] and [15]. From many other systems for this purpose is usually chosen as simple system as possible and the application is very often taken from the biology, mainly due to relative easy possibility of verification starting considerations.

The most simple population model would assume a constant growth rate, but in that situation we find unlimited growth, which is not realistic. In our model we will assume that the population is restricted by a constant environment but this premise requires a modification of the growth law. Now the growth rate depends on the actual size of the population relative to its maximal size.

What is a population dynamics model actually? It is simply a law, which is characteristic for some biological species and allows us to predict the population development of that species in time. Time is measured in increments t = 0, 1, 2. (Minutes, hours, days, years, whatever is appropriate). The size of the population is measured at time *t* by the actual number of the species *x(t)*,.

Naturally, the size of a population may depend on many parameters such as environmental conditions (e.g., food supply, space, climate), interaction with other species (e.g., the predator/prey relationship), but also age structure, fertility, etc. The complexity of influences that determine a given population in its growth behaviour will be shown in following pictures.

2 Verhulst model of dynamics population

The observation of the Belgian mathematician Pierre Francois Verhulst and his work around 1845 leads to the postulate that the growth rate at time *t* should be proportional to the difference between the population count and the maximal population size, which is a convenient measure for the fraction of the environment that is not yet used up by the populations at time *t*. This assumption leads to the following population model [3]:

$$\frac{dx(t)}{dt} = c * x(t) * \left(1 - x(t)\right) \qquad (1)$$

Where *x (t)* measures the relative population count *x(t) = X(t)/N* and *N* is the maximal population size which can be supported by the environment, *c* denote suitable constant. The model is continuous in time, but a modification of the continuous equation to a discrete quadratic recurrence equation known as the logistic map is also widely used.
Now let we try to use common method for the solution of the differential equation (1), for example well known Runge - Kutta 4 method, from the software package MATHCAD 2001. For the solution was used the *c=2,8* and the initial value *x (0) =0,5*. Number of calculation steps was 100. The solution is in the Fig.1, where x-axes denote time, and y-axes denote of the relative population count *x (t)*. For the other experiments we can use solving of the equation (1) by Euler method putting the difference instead of the differential symbol.

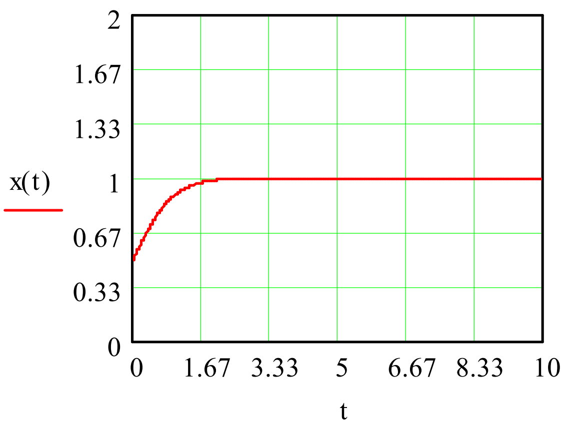

Fig.1 The equation (1) solving for the *c=2,8* by Runge Kutta 4 method

$$\frac{dx(t)}{dt} \approx \frac{x(t+1)-x(t)}{\Delta t} \qquad (2)$$

Connecting (1) and (2) we obtain:

$$\frac{x(t+1)-x(t)}{\Delta t} = c * x(t) * (1-x(t)) \qquad (3)$$

After small manipulation one can obtain Verhulst population model in the form so called logistic equation (4)

$$x(t+1) = a * x(t) * (1-x(t)) + x(t) \qquad (4)$$

where the suitable constant now has form *a=c*Δt.*

3 Results of experiments with solving logistic equation by iteration method

For solving equation (4) is possible to use iteration method for the initial condition *x(0) = 0,5* and the constant a let varying following manners $2 \leq a \leq 3$. In the series of figures one can see how will be changing the solution for the selected value of the constant *a*.

This solution and the previous one for the time $t \to \infty$ has the same value as the solving equation (1) by method Runge Kutta 4. Now we try to move value of constant *a* from the value *a=2* to *a=3*, even if this change is seemed to be small, but the solution give different value from previous one.

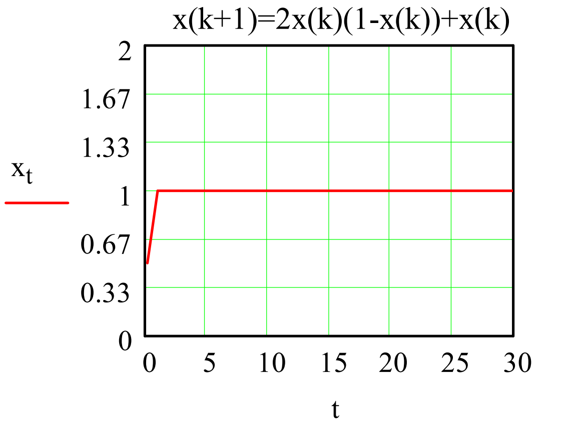

Fig.2 The logistic equation solving for the constant *a=2*

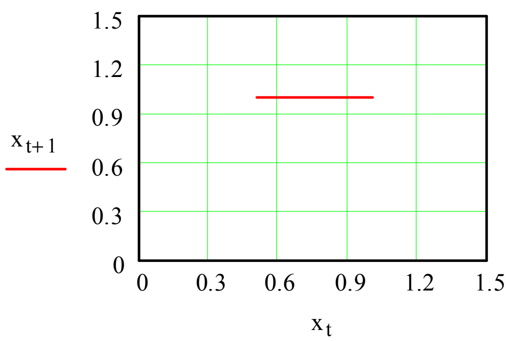

Fig. 3 The phase portrait for logistic equation solved in the fig. 2.

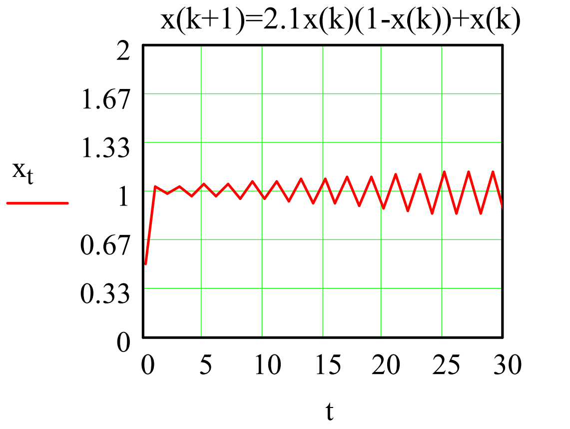

Fig. 4 The logistic equation solving for the constant *a=2,1*

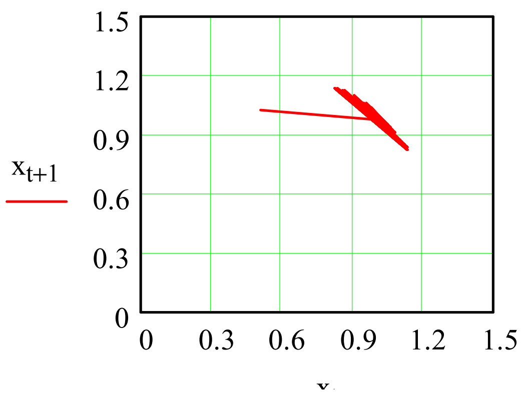

Fig. 5 The phase portrait for logistic equation solving for the constant *a=2,1*

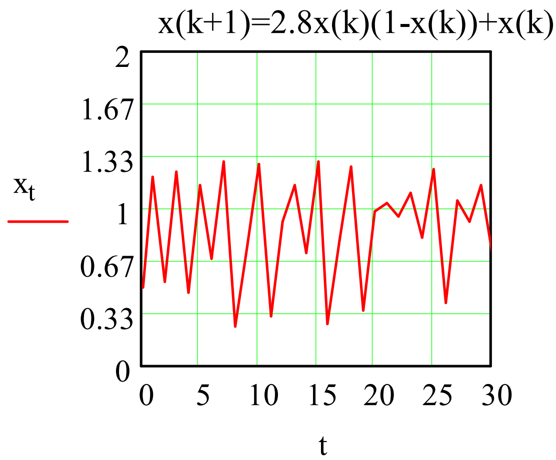

Fig. 6 The logistic equation solving for the constant *a=2,8*

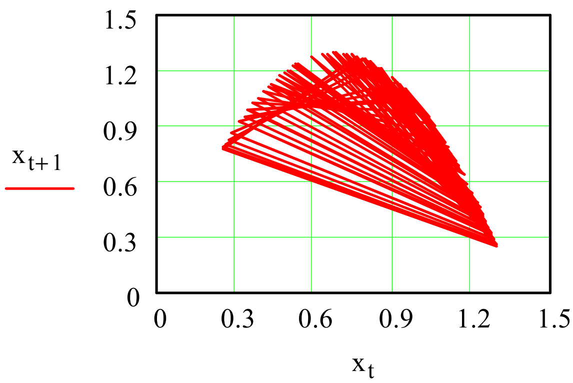

Fig. 7 The phase portrait for logistic equation solving for the constant *a=2,8*

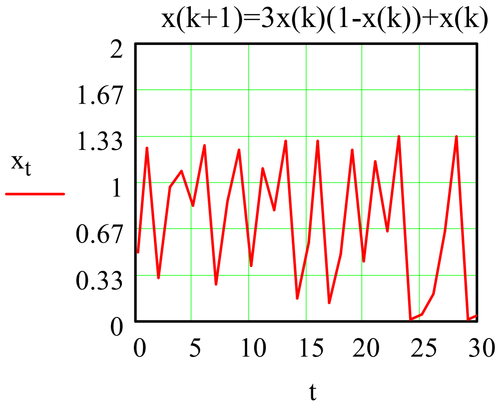

Fig. 8 The logistic equation solving for the constant *a=3*

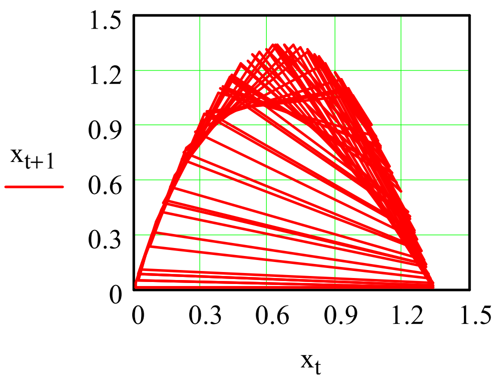

Fig. 9 The phase portrait for logistic equation solving for the constant *a=3*

The chaos behaviour starting from the value *a=2,01*, (see fig. 13) but visible significant changing in phase portrait starting from value *a=2,02* in compare with fig. 3. This situation is on the following picture Fig. 10.

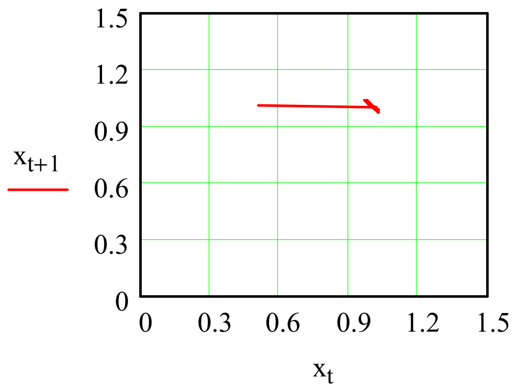

Fig. 10 The phase portrait for logistic equation solving for the constant *a=2,02*

But the time dependent picture is still without visible changing, they look like fig. 2. If we increasing the constant *a* to higher values, we obtain even much small even much higher values.

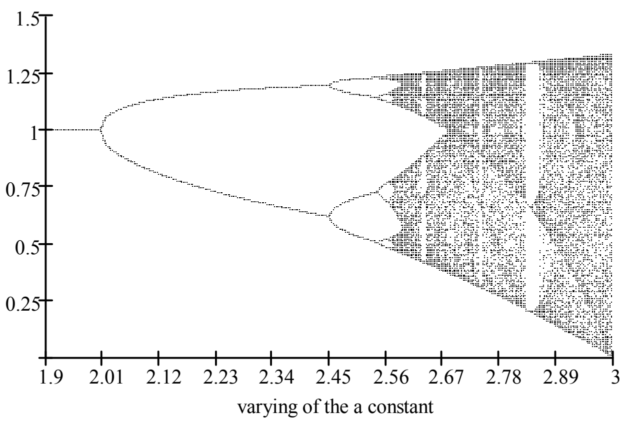

Fig. 12 Bifurcation of the equation 4.

4 Conclusions and future work

If we compare solutions of the Verhulst dynamic model with the same initial conditions population, one solved by Runge-Kutta method and the other by Euler's method, i.e. substitute derivation by means of difference (see equation (2), (3), and (4)), we obtain quite different solutions. What happened?

In equation (4) was supposed *Dt=1*, and for this assumption was made previous solving which generate chaotic behaviour of the system described by equation (4).

If within equation (4) $Dt^{1}1$, for example *Dt=0,4*, *c=2,8*, so that *a=0,4*2,8=1,12* (see fig. 11).

It is obvious that solution is similar to one on the fig. 1.

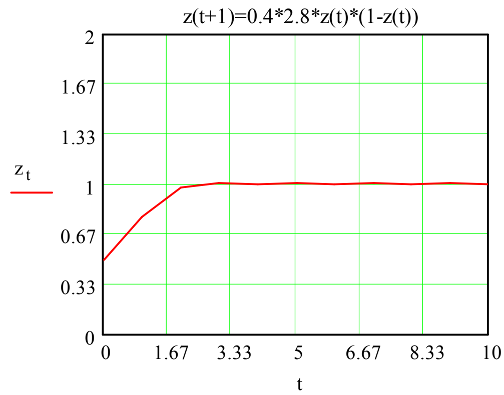

Fig. 11 Solving the equation (4) for *a=0,4*2,8=1,12*

If the Runge Kutta 4 method is used for solving equation (1) with following parameters (*c=2.8, init_vals=0.5, init_x=0, final_x=20, Nsteps=15,* $Dx(t)=cX_0(1-X_0)$) it gave chaotic behaviour.

One can see that the parameter *Nsteps* was dramatically decreased (normal value is about *Nsteps=100* and more), so that the computation is not so exact.

Runge Kutta 4 formula in MATHCAD 2001 package:

$$R = \mathrm{rkfixed}(\mathrm{int\_vals},\mathrm{init\_x},\mathrm{final\_x},\mathrm{Nsteps},\mathrm{Dx}) \quad (5)$$

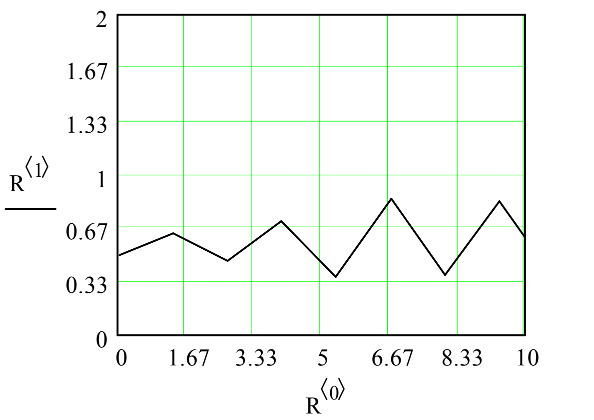

Fig. 13 Equation (1) solved by Runge–Kutta 4 method with dramatically decreasing parameter *Nsteps.*

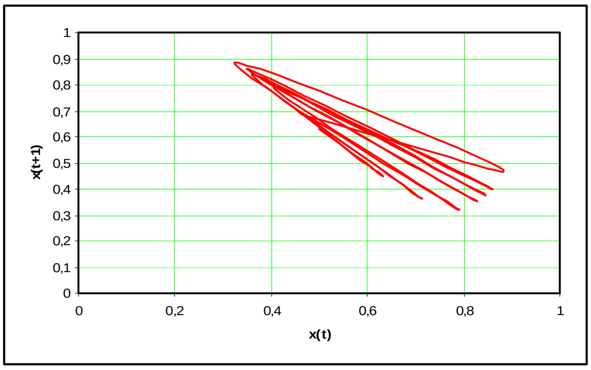

Fig. 14. The phase portrait of equation solving done at fig. 13.

If we use the numerical method like Runge Kutta 4 (with usual value of *Nsteps* parameter) or standard form for differential equation solving, we will never thing about chaotic behaviour of the system described by Verhulst model.

Is the chaotic behaviour of this system made by numerical solution or used numerical method? Does the system have chaotic behaviour itself or such behaviour has only the mathematical model of the system? For these questions is difficult to find right answers. It seems that the chaotic behaviour of the system described by means of the equation (1) or (4) depend on the precision of the numerical method and sensitivity of this model on initial conditions as is more generally described in [8].

The future works should lead to find other simple systems for better understanding its behavior in connection of their models numerical solution.

Note:

In the most literature sources is under logistic function (logistic map) thought following version:

$$x(t+1) = a * x(t) * (1 - x(t)) \tag{6}$$

For this version (6) the constant *a* varying from value 2,5 to 4, and chaotic behaviour start with value *a=3* to *a=4.* If we compare this results with fig. 14, the chaotic behaviour in this case (4) starting with point *a=2.01* till *a=3.*

References:

[1] Kaye, B.: *Chaos & Complexity*. VCH, New York, 1993

[2] Peitgen at all.: *Fractals for the classroom, part one.* Springler Verlag, New York, 1992

[3] Mandl,P.: *Pravděpodobnostní dynamické modely.* Academia, Praha, 1985

[4] Gleick,J.: *Chaos: Making a New Science* (Czech translation) Ando Publishing, Brno 1996

[5] Leibovitch L,S.: *Fractals and Chaos Simplified for the Life Science.*Oxford University Press 1998

[6] Puu,T.: *Attractors, Bifurcations,& Chaos*. Springer-Verlag Berlin Heidelberg 2003

[7] Conveney,P. Highfield, P.: *Frontiers of Complexity*(Czech translation) Mladá Fronta 2003

[8] Jura P.: Some remarks on mathematical models. *WSEAS Transactions on Information Science and Applications. Issue 5, Vol. 1, Nov. 2004, pp1426-1429*

[9] http://folk.ntnu.no/berland/math/feigenbaum/ index.html

[10] http://www.cut-the-knot.org/blue/chaos.shtml

[11] http://www-solar.mcs.stand.ac.uk /~alan/MT2003/Numerical/node13.html

[12] Wikipedia contributors,'Logistic function', *Wikipedia, The Free Encyclopedia,*15July 2006, 09:59 UTC, <http://en.wikipedia.org/w/index.php?title=Logistic_function&oldid=63933823> [accessed 16 July 2006]

[13] http://mathworld.wolfram.com

[14] Čapek,J.: Chaotic behaviour of systems some notes. In: *Process Control 2000, University of Pardubice, Pardubice, 2000, 5p, ISBN 80-7194-271-5.*

[15] Čapek J.: Verhulst model of population In.: Riadenie a informatika v novom tisícročí, *MI 2000 Žilinská Univerzita v Žiline pp321-366*, ISBN 80-7100-757-9