Numerical Method for a Non-Homogeneous Equation of Stationary Waves

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Abstract: In this paper we present an algorithm for solving a Dirichlet problem for the stationary equation of the form $-\Delta\varphi(x,y)+k^2\varphi(x,y)=-f(x,y),\ (x,y)\in D=[0,1]\times[0,1]$. The numerical examples allow a detailed analysis of the problem solution, which depends of the form of f, the value of k and of the step h of the network defined on D.

Key words: Helmholtz equation, periodical solution, finite difference methods, Fast Fourier Transform (FFT), interpolation polynomial.

1. Introduction

Let us consider a rectangular diaphragm D with fixed boundary Γ, which vibrates under the action of periodical forces g

$$g=-\frac{\mu_1(x,y)}{c^2}\cos\omega t+\frac{\mu_2(x,y)}{c^2}\sin\omega t \qquad (1)$$

The boundary value problem for wave equation is

$$\Delta u(x,y,t)-\frac{1}{c^2}\frac{\partial^2 u(x,y,t)}{\partial t^2}=g(x,y,t) \qquad (2)$$
$$\forall(x,y)\in D\subset \mathbf{R}^2$$

$$u\big|_\Gamma=0 \qquad (3)$$

where c is a constant and the boundary Γ of D is a smooth contour. The function $u(x,y,t)$ in the initial moment will not be periodical. After any time will appear the periodic oscillations with the same frequency ω of the external forces g, such that

$$u(x,y,t)=\varphi(x,y)\cos\omega t-\psi(x,y)\sin\omega t \qquad (4)$$

where ω is a given number, which is called the frequency of periodic oscillations and is equal to the number of oscillations in 2π units of time.
Substituting (1) and (4) in (2), we get the following equations in view of linear independence of the functions $\cos\omega t$ and $\sin\omega t$

$$\Delta\varphi(x,y)+k^2\varphi(x,y)=f(x,y) \qquad (4)$$
$$\varphi\big|_\Gamma=0 \qquad (5)$$

and

$$\Delta\psi(x,y)+k^2\psi(x,y)=h(x,y) \qquad (6)$$
$$\psi\big|_\Gamma=0 \qquad (7)$$

where $f=-\frac{\mu_1}{c^2},\ h=-\frac{\mu_2}{c^2},\ k=\frac{\omega}{c}.$

For the problem (4)-(5), we find the values of φ in the points of a network defined over the domain D, using the finite differences schema and the Fast Fourier Transform(FFT) to calculate the coefficients of a trigonometric interpolation polynomial of f. Then, a interpolation of the reticular function φ is made. The algorithm is verified by numerical examples.

This method is an extension of the theoretical results which was obtained by Cooley, Tuckey [8], Marciuk [5] and Rombaldi [7], for the stationary problems. Unlike, the other papers, the numerical examples prove that this algorithm is flexible and computationally efficient for the solution of an elliptic equation.

2. Problem formulation

Let us consider a Dirichlet problem for a non-homogeneous Helmholtz equation (4). To use the method of finite differences, it is necessary that the domain D = [0, 1]×[0, 1] to be replaced by a quadratic network with the step $h = 1/N$, and nodes $x_k = k/N,\ y_l = l/N$. Hence

$$D_N = \{(x_k, y_l) \mid 0 \le k \le N,\ 0 \le l \le N\}.$$

Let Γ be the set of nodes of the form (x_0, y_l), (x_l, y_0), (x_N, y_l), (x_l, y_N) and let function $\varphi(x)$ continuous on D be the solution of (4) - (5). This is approximated by a reticular function φ^h. Defining $\varphi^h_{k,l} = \varphi(x_k, y_l)$, the Laplacian operator in node (x_k, y_l) is of the form

$$(\Delta^h \varphi^h)_{k,l} = \frac{\varphi^h_{k+1,l} + \varphi^h_{k-1,l} + \varphi^h_{k,l+1} + \varphi^h_{k,l-1} - 4\varphi^h_{k,l}}{h^2}$$

and $\varphi^h_{k,l} = 0$ on Γ. (8)

Once the step h was established we denote the reticular function φ^h with φ and problem (8) becomes

$$\frac{4\varphi_{k,l} - \varphi_{k-1,l} - \varphi_{k+1,l} - \varphi_{k,l-1} - \varphi_{k,l+1}}{h^2} + \mu\varphi_{k,l} = f(x_k, y_l) \qquad (9)$$

$$\varphi(x_k, x_l) = 0 \text{ on } \partial D,\ k, l \in \{0,1,\dots,N\}. \qquad (10)$$

Introducing the notations

$$\varphi_l = \begin{bmatrix} \varphi_{1,l} \\ \varphi_{2,l} \\ \vdots \\ \varphi_{N-1,l} \end{bmatrix},\ f_l = \begin{bmatrix} f_{1,l} \\ f_{2,l} \\ \vdots \\ f_{N-1,l} \end{bmatrix}, \qquad (11)$$

$l = 1,2,\dots,N-1$ and defining the matrix of $(N - 1)$ order

$$B = \begin{bmatrix} h^2k^2-4 & 1 & 0 \cdots 0 & 0 & 0 \\ 1 & h^2k^2-4 & 1 \cdots 0 & 0 & 0 \\ \vdots & \vdots & \vdots \cdots \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 1 & h^2k^2-4 & 1 \\ 0 & 0 & 0 \cdots 0 & 1 & h^2k^2-4 \end{bmatrix}$$

we obtain the following set of equations

$$\begin{cases} B\varphi_1 - \varphi_2 = h^2 f_1 \\ -\varphi_{l-1} + B\varphi_l - \varphi_{l+1} = h^2 f_l, l = 2,\dots,N-2 \\ -\varphi_{N-2} + B\varphi_{N-1} = h^2 f_{N-1} \end{cases} \qquad (12)$$

The solutions of the eigenvalues problem

$$Bv^{(m)} = \lambda_m(B)v^{(m)}$$

is of the form

$$\lambda_m(B) = -2\left(1 + \cos\frac{(N-m)\pi}{N}\right) + h^2k^2 - 2 \qquad (13)$$

m = 1, 2, …, N-1

and

$$v^{(m)}_k = (-1)^k \sqrt{\frac{2}{N}} \sin\frac{m\pi k}{N},\ k = 1,2,\dots, N\text{-}1 \qquad (14)$$

are the components of the vector $v^{(m)}$.

Since the eigenvalues of the matrix B are distinct, the corresponding eigenvectors are linearly independent. The factor $\sqrt{\frac{2}{N}}$ has been chosen since the basis is orthonormal in the space $\mathbf{R}^{N-1}$.

Now we verify this property of the vectors $\{v^{(m)}\}$, m = 1,2,.., N-1. In $\mathbf{R}^{N-1}$ space the product scalar of the two vectors is of the form

$$(v, w) = \sum_{k=0}^{N-1} v_k w_k,\ v_0 = v_N = 0 \qquad (15)$$

Then, for $m \ne r$, we have

$$(v^{(m)}, v^{(r)}) = \frac{2}{N}\sum_{k=0}^{N-1}(-1)^{m+r} \sin\frac{mk\pi}{N} \sin\frac{rk\pi}{N} =$$

$$= \frac{(-1)^{m+r}}{N}\sum_{k=0}^{N-1}\cos\frac{(m-r)k\pi}{N} - \frac{(-1)^{m+r}}{N}\sum_{k=0}^{N-1}\cos\frac{(m+r)k\pi}{N}.$$

But

$$\sum_{k=0}^{N-1}\cos\frac{lk\pi}{N} = \frac{1}{2}\sum_{k=0}^{N-1}\left(e^{i\frac{lk\pi}{N}} + e^{-i\frac{lk\pi}{N}}\right) =$$

$$= \frac{1}{2}\frac{1-e^{il\pi}}{1-e^{i\frac{l\pi}{N}}} + \frac{1}{2}\frac{1-e^{-il\pi}}{1-e^{-i\frac{l\pi}{N}}} = \begin{cases} 0, & \text{if } l \text{ even} \\ 1, & \text{if } l \text{ odd} \end{cases}.$$

Then, for $m \neq r$, $(v^{(m)}, v^{(r)}) = 0$ and for $m = r$

$$(v^{(m)}, v^{(m)}) = \frac{1}{N}\sum_{k=0}^{N-1}\cos 0 - \frac{1}{N}\sum_{k=0}^{N-1}\cos\frac{2mk\pi}{N} =$$

$$= h \cdot N - h \cdot 0 = 1.$$

In space $\mathbf{R}^{N-1}$ the vectors f_l , l = 1,2,…, N-1 are represented as:

$$f_l = \sum_{m=1}^{N-1} F_{m,l} v^{(m)} \tag{16}$$

or

$$f_{l,k} = \sum_{m=1}^{N-1} F_{m,l} v_k^{(m)} \tag{17}$$

where

$$F_{m,l} = \sqrt{\frac{2}{N}}\sum_{k=1}^{N-1} f_{l,k} \sin\frac{2mk\pi}{2N} \tag{18}$$

It should be observed that

$$F_{m,l} = (f_l, v^{(m)}) \tag{19}$$

and

$$(f_l, f_l) = \sum_{m=0}^{N-1} F_{m,l}^2 \tag{20}$$

The sum (16) is the finite Fourier series for the function f_l and (20) is Parseval's formula.

Now we assume that, the unknown functions are represented as

$$\varphi_l = \sum_{m=1}^{N-1} \Phi_{m,l} v^{(m)}, \tag{21}$$

Using (16) and (21) in (12) and multiplying with the vector $v^{(m)}$, one obtains for each fixed λ_m a system of equations with a band coefficients matrix

$$\begin{cases} \lambda_m \Phi_{m,1} - \Phi_{m,2} = F_{m,1} \\ -\Phi_{m,l-1} + \lambda_m \Phi_{m,l} - \Phi_{m,l+1} = F_{m,l}, \\ \qquad l = 2,\dots, N-2 \\ -\Phi_{m,N-2} + \lambda_m \Phi_{m,N-1} = F_{m,N-1} \end{cases} \tag{22}$$

We remind that, for a periodic function f having the period T_x=1, the interpolation polynomial is of the form

$$f_j = f(x_j) = P(x_j) = \frac{\alpha_0}{2} + \frac{\alpha_N}{2}\cos 2\pi N x + \sum_{k=1}^{N-1}\left(\alpha_k \cos 2\pi k x_j + \beta_k \sin 2\pi k x_j\right) \tag{23}$$

where

$$\alpha_k = \frac{1}{N}\sum_{j=0}^{2N-1} f_j \cos 2\pi k x_j, \quad \beta_m = \frac{1}{N}\sum_{j=0}^{2N-1} f_j \sin 2\pi m x_j,\ x_j = \frac{j}{2N}, \tag{24}$$

$k \in \{0,\dots,N\}$, $m \in \{1,\dots,N-1\}$, $j \in \{0,1,\dots,2N-1\}$.

We observe that the relations (18) differ from the Fourier coefficients β_j by a constant. Using (18) and (24), we get for $h^2 f_l$ that

$$F_{m,l} = \frac{1}{N}\sqrt{\frac{2}{N}}\,\beta_m \tag{25}$$

for each fixed l.

Let us described the method for the estimation of $\beta_{j,l}$ coefficients using FFT, [3], [5], [7]. In order to this, a network on [0,1] with 2N nodes was considered for each fixed l and $f_{0,l} = f_{N,l} = \dots = f_{2N-1,l} = 0$. To halve the computations number one separates the components of $f_{j,l}$ with even index from those with odd index for each l. Let

$$u_j = f_{2j} + i.f_{2j+1},\ j = 0, 1,\dots, N-1 \tag{26}$$

and define *Direct Discret Fourier Transform* (DDFT)

$$u_j = \sum_{k=0}^{N-1} c_k v_k^j \tag{27}$$

with $v_k^j = \exp\left(i\,\frac{2\pi k j}{N}\right)$, $j \in \{0,1,\dots, N\text{-}1\}$. DDFT can be expressed matriceally as:

$$U = T \cdot C$$

where

$U = (u_0, u_1, \dots, u_{N-1})^t$; $C = (c_0, c_1, \dots, c_{N-1})^t$

and t – the operation of transposition.

$$T = (v_k^j) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & v_1 & v_2 & \cdots & v_{N-1} \\ 1 & v_1^2 & v_2^2 & \cdots & v_{N-1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & v_1^{N-1} & v_2^{N-1} & \cdots & v_{N-1}^{N-1} \end{bmatrix},$$

where $v = e^{\frac{2\pi i}{N}}$.

Since $|v| = 1$ and $\bar{v} = e^{-i\frac{2\pi}{N}} = 1/v$ one obtains $T\bar{T} = \bar{T}T = NI_N$, where $\bar{T}$ is the conjugate matrix of T. It follows from the form of T that it is a nonsingular matrix so that $T^{-1} = \frac{1}{N}\bar{T}$. Thus, using (27) one obtains

$$c_j = \frac{1}{N}\sum_{k=0}^{N-1} v_k^{-j} u_{k,j},\ j \in \{0,1,\dots,N-1\} \qquad (28)$$

which define the Inverse Discrete Fourier Transform and they can be written in a matrix form as

$$C = \frac{1}{N}\bar{T}U \qquad (29)$$

Considering that $v_k^{-j} = v_k^{N-j}$ and (29) we get

$$C = \frac{1}{N}PTU \qquad (30)$$

where P is a permutation matrix.

With the next theorem,[1], we find the coefficients β_m from (30) and (25) for every l.

Theorem, [8].
Let D_{2N} be a network on the interval [0, 1], $\{f_k\}$, $k\in\{0, 1, \dots, 2N-1\}$ be the reticular function, where $N = 2^p$, $p\in\mathbf{N}^*$ and u_k defined by (27). Then the coefficients of trigonometrically polynomial (23) are computed from the formulas

$$\alpha_j - i\beta_j = \frac{1}{2}(c_j + \bar{c}_{N-j}) - -\frac{i}{2}(c_j - \bar{c}_{N-j})\exp(-i\frac{\pi j}{N}),\ \beta_0 = \beta_N = 0; c_N = c_0$$
$$\alpha_{N-j} - i\beta_{N-j} = \frac{1}{2}(c_{N-j} + \bar{c}_j) - -\frac{i}{2}(\bar{c}_j - c_{N-j})\exp(i\frac{\pi j}{N}),\quad j = 0,1,\dots\dots,N. \qquad (31)$$

In general, the complex products number from FFT algorithm has value $N\log_2 N$. If u is not defined by (26), the number of the computations decreases if the matrix T of NxN dimensions is replaced by a product of sparse matrices.

After we determine $F_{m,l}$, $m, l = 1, 2, \dots, N-1$ with (25), we will solve (22) for each λ_m and from (21) we get

$$\varphi_{jl} = \sum_{m=1}^{N-1}\Phi_{m,l}v_j^{(m)},\ j,l = 1,2,\dots,N-1 \qquad (32)$$

3. Numerical examples

I. Let's define a quadratic network with $N = 4$, $h = 1/4$, $k^2 = 1$ and a periodic function $f(x, y) = (1-2\pi^2)\sin\pi x\sin\pi y$ with period $T_x = 1$. In this case, the matrix B from (12) is of the form

$$B = \begin{bmatrix} -3.94 & 1 & 0 \\ 1 & -3.94 & 1 \\ 0 & 1 & -3.94 \end{bmatrix}$$

and the eigenvalues have the values

$$\lambda_1 = -2.523,\ \lambda_2 = -3.94,\ \lambda_3 = -5.35 \qquad (33)$$

Generally, $\lambda_{N/2}$ is arithmetic mean of the others eigenvalues.

Let us now find the eigenvector components corresponding to eigenvalues λ_m with $m = 1, 2, 3$. These are

$$v_1^{(m)} = \frac{-1}{\sqrt{2}}\sin\frac{\pi m}{4},\ v_2^{(m)} = \frac{1}{\sqrt{2}}\sin\frac{2\pi m}{2},\ v_3^{(m)} = \frac{-1}{\sqrt{2}}\sin\frac{3\pi m}{4} \qquad (34)$$

and (22) will be

$$\begin{cases} \lambda_m\Phi_{m,1} - \Phi_{m,2} = F_{m,1} \\ -\Phi_{m,1} + \lambda_m\Phi_{m,2} - \Phi_{m,3} = F_{m,2} \\ -\Phi_{m,2} + \lambda_m\Phi_{m,3} = F_{m,3} \end{cases} \qquad (35)$$

where we have successively $m = 1, 2, 3$.

To obtain the solutions of (35), the Fourier coefficients of the vectors f_l, $l\in\{1, 2, 3\}$ have to be computed. For each fixed l we use $2N = 8$ nodes on [0,1], $x_j = j/2N$, $j = 0,1,\dots, 2N-1$ and in this aim we define the matrix

$$[f_0 f_1 \dots\dots f_7]_l^T = [0 \quad f_{1,l} \quad \cdots \quad f_{3,l} \quad 0 \quad 0 \quad 0 \quad 0]^T,$$

$$U = \begin{bmatrix} i\cdot f_{1,l} \\ f_{2,l} + i\cdot f_{3,l} \\ 0 \\ 0 \end{bmatrix}. \qquad (36)$$

In our case, the distribution of $f_{k,l}$, $k, l = 1, 2, 3$ is

Tabel 1

x_k	0	1/4	1/2	3/4
$l = 1$ ($y = 1/4$)	0	-9.36	-13.24	-9.36
$l = 2$ ($y = 1/2$)	0	-13.24	-18.72	-13.24
$l = 3$ ($y = 3/4$)	0	-9.36	-13.24	-9.36

Since $\nu_k^{-j} = \nu_k^{N-j}$, the permutation matrix is obtained as

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix} \qquad (37)$$

and using (30) we obtain for $l = 1$

$$C = \frac{1}{4} PTU =$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \nu_0^0 & \nu_1^0 & \nu_2^0 & \nu_3^0 \\ \nu_0^1 & \nu_1^1 & \nu_2^1 & \nu_3^1 \\ \nu_0^2 & \nu_1^2 & \nu_2^2 & \nu_3^2 \\ \nu_0^3 & \nu_1^3 & \nu_2^3 & \nu_3^3 \end{bmatrix} \cdot \begin{bmatrix} -9.36i \\ -13.24 - 9.3i \\ 0 \\ 0 \end{bmatrix} =$$

$$= \frac{1}{4} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \cdot \begin{bmatrix} -9.36i \\ -13.24 - 9.3i \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$C = \frac{1}{4} \cdot \begin{bmatrix} -3.3 - 4.7i \\ -2.33 + 0.97i \\ 3.3 \\ 2.34 - 5.65i \end{bmatrix}$$

From (31) and (25) we find
for $l = 1$:
$\beta_0 = 0$, $\beta_1 = -6.62$, $\beta_2 = 0$, $\beta_3 = 0$ and then
$F_{1,1} = -1.17$, $F_{2,1} = 0$, $F_{3,1} = 0$;

for $l = 2$:

$$F_{1,2} = -1.654, \quad F_{2,2} = 0, \quad F_{3,2} = 0 \qquad (38)$$

for $l = 3$:
$F_{1,3} = -1.17$, $F_{2,2} = 0$, $F_{3,2} = 0$.

Now we can verify Parseval's formula for $l = 1$:

$$\sum_{k=1}^{3} (h^2 f_{k,1})^2 = \sum_{m=1}^{3} F_{m,1}^2 \qquad (39)$$

Indeed, $\frac{1}{4^4}\left(9.36^2 + 13.24^2 + 9.36^2\right) = 1.17^2$.

We will identify the solutions of the system (35) for $m \in \{1, 2,3\}$ with λ_m from (33). The values $F_{m,k}$ lead to only matrix equation, which corresponds to $\lambda_1 = -2.52$:

$$\begin{bmatrix} -2.53 & -1 & 0 \\ -1 & -2.53 & -1 \\ 0 & -1 & -2.53 \end{bmatrix} \cdot \begin{bmatrix} \Phi_{1,1} \\ \Phi_{1,2} \\ \Phi_{1,3} \end{bmatrix} = \begin{bmatrix} 1.17 \\ -1.654 \\ 1.17 \end{bmatrix}$$

in accordance to the signs of eigenvectors. From (21) and (34), the approximate solution φ and the exact solution, $\varphi_e = \sin(\pi x)\sin(\pi y)$ are presented in the tables 2

Table 2

x_k	1/4		1/2		3/4	
	φ	φ_e	φ	φ_e	φ	φ_e
l =1	0.53	0.5	0.74	0.71	0.53	0.5
l =2	0.75	0.71	1.06	1	0.75	0.71
l =3	0.53	0.5	0.74	0.71	0.53	0.5

II. Now will be considered the same example, but with $N = 2^3$. The eigenvalues are: $\lambda_1 = -2.14$, λ_2=-2.6, λ_3=-3.2, λ_4=-3.9, λ_5=-4.7, λ_6=-5.4, λ_7=-5.8 and the numerical solution is presented in table 3.

Table 3

k		1	2	3	4	5	6	7
$\varphi_l\,(x_k)$	l=1	0.148	0.274	0.358	0.388	0.358	0.274	0.148
	l=2	0.274	0.506	0.661	0.716	0.661	0.506	0.274
	l=3	0.358	0.661	0.864	0.936	0.864	0.661	0.358
	l=4	0.387	0.716	0.936	1.013	0.936	0.716	0.387
	l=5	0.358	0.661	0.864	0.936	0.864	0.661	0.358
	l=6	0.274	0.506	0.661	0.716	0.661	0.506	0.274
	l=7	0.148	0.274	0.358	0.388	0.358	0.274	0.148

III. Let us now consider the different values for constant k^2.

a. If k^2 = 10, the eigenvalues are:
$\lambda_1 = -1.96,\ \lambda_2 = -3.8,\ \lambda_3 = -5.23$
and the numerical solution

Table 4

x_k	1/4	1/2	3/4
l=1	0.55	0.786	0.55
l=2	0.786	1.112	0.786
l=3	0.55	0.786	0.55

b. If k^2 = 100, the eigenvalues are:
$\lambda_1 = 3.66,\ \lambda_2 = 2.25,\ \lambda_3 = 0.836$
and the numerical solution

Table 5

x_k	1/4	1/2	3/4
l=1	0.494	0.698	0.494
l=2	0.698	0.988	0.698
l=3	0.494	0.698	0.494

IV. If in the first case on replaced f with

$$f(x,y) = xy(1-x)(y-1) + 2(x+y-x^2-y^2)$$

we obtain the approximate solution (k = 1):

Table 6

x_k	1/4	1/2	3/4
l=1	-0.034	-0.045	-0.035
l=2	-0.048	-0.064	-0.048
l=3	-0.034	-0.045	-0.034

In this case, the exact solution is of the form: $\varphi_e = = xy(1-x)(y-1)$ and in the point of network it has the values

Table 7

x_k	1/4	1/2	3/4
l=1	-0.035	-0.047	-0.035
l=2	-0.047	-0.063	-0.047
l=3	-0.035	-0.047	-0.035

6. CONCLUSIONS

If we denote $g(x) = \varphi(x,y_l)$ for fixed l and $x\in[0,1]$, then it may be expanded in a trigonometric Fourier series at any point x of this interval:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx) \quad (40)$$

where

$$a_n = 2\int_0^1 g(x)\cos 2\pi nx dx,\ \ b_n = 2\int_0^1 g(x)\sin 2\pi nx dx .$$

Let us now consider $P_N(x)$, the Fourier interpolation polynomial for g and we get

$$g(x_i) = P_N(x_i),\ x_i = \frac{i}{N}, i = 0,1,\ldots,N,$$
$$g(x_0) = g(x_N)$$

where

$$P_N(x) = \frac{A_0}{2} + \sum_{k=1}^{N-1}\left(A_k \cos 2\pi kx + B_k \sin 2\pi kx\right).$$

The increase of nodes number N of the network leads to a good approximation of the solution, which may be verified with discrete Parseval's equality

$$\frac{1}{4}A_0^2 + \frac{1}{2}\sum_{n=1}^{N-1}(A_n^2 + B_n^2) = \frac{1}{2N}\sum_{i=0}^{2N-1} g^2(x_i)$$

When N increases, the sum from the right part tends to $\int_0^1 g^2(x)dx$. If the integral is bounded, the sum is bounded. Therefore, the sum of the left converge and A_n, B_n will be small. Thus, the trigonometric interpolation will converge for continuous function g or for function g with a finite number of points of discontinuity.

The numerical examples were obtained with the help of a program written in MathCAD. We show that the increasing of the k value leads to the increasing of the eigenvalues. Also, we find how depends the sign of the eigen-values of the k . Besides, we observe that the error, $e = \varphi - \varphi_e$, has negative value for positive eigenvalues and positive value for negative eigenvalues.

As before, it follows from the analysis of numerical results that the error tends to zero with increasing N. The numerical results prove that the errors are smaller when the periodical force is a polynomial, than it is a periodical trigonometric force.

References

[1]. Brigham. E. O., The Fast Fourier Transform, Prentice-Hall, Englewood Cliffs, 1974.
[2] Dang Q., A approximate method for solving an elliptic problem with discontinuous coefficients, J. Comput. Appl.Math., vol.51, No.2,1994, pp.193-203.
[3] Godunov S.K., Reabenki V.S., The theory of difference schemes, Editura Tehnica, Bucharest, 1977.

[4] Marciuk G. I., Shayduov V., Refining of the solutions of the difference schemes, Mir Publishers, Moscow, 1983.
[5] Marciuk G. I., Numerical Methods, Edit. Academiei, Bucuresti, 1983.
[6] Mastorakis N. and Martin O., On the solution of integral-differential equations via the Rayleigh-Ritz finite elements method, *WSEAS Transactions on Mathematics,* Vol.4, No.2, 2005.
[7] J. E. Rombaldi*, Problèmes corrigés d'analyse numérique* (Masson, Paris, 1996).
[8] J. Stoer, *Einführung in die Numerische Mathematik I* (Springer - Verlag, Berlin, 1972).
[9] Tihonov, A. and Samarskii,A, *Equations of Mathematical Physics,* Dover Publ., New York, 1990.