Numerical Method for a
Non-Homogeneous Equation of Stationary Waves

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Abstract: In this paper we present an algorithm for solving a Dirichlet problem for the stationary equation of the form
\[-\Delta \varphi(x, y) + k^2 \varphi(x, y) = -f(x, y), (x, y) \in D = [0, 1] \times [0, 1].\]
The numerical examples allow a detailed analysis of the problem solution, which depends on the form of \(f\), the value of \(k\) and of the step \(h\) of the network defined on \(D\).

Key words: Helmholtz equation, periodical solution, finite difference methods, Fast Fourier Transform (FFT), interpolation polynomial.

1. Introduction

Let us consider a rectangular diaphragm \(D\) with fixed boundary \(\Gamma\), which vibrates under the action of periodical forces \(g\)

\[g = -\frac{\mu_1(x, y)}{c^2} \cos \omega t + \frac{\mu_2(x, y)}{c^2} \sin \omega t\]  \hspace{1cm} (1)

The boundary value problem for wave equation is

\[\Delta u(x, y, t) - \frac{1}{c^2} \frac{\partial^2 u(x, y, t)}{\partial t^2} = g(x, y, t)\]  \hspace{1cm} (2)

\[\forall (x, y) \in D \subset \mathbb{R}^2\]

\[u|_{\Gamma} = 0\]  \hspace{1cm} (3)

where \(c\) is a constant and the boundary \(\Gamma\) of \(D\) is a smooth contour. The function \(u(x, y, t)\) in the initial moment will not be periodical. After any time will appear the periodic oscillations with the same frequency \(\omega\) of the external forces \(g\), such that

\[u(x, y, t) = \varphi(x, y) \cos \omega t - \psi(x, y) \sin \omega t\]  \hspace{1cm} (4)

where \(\omega\) is a given number, which is called the frequency of periodic oscillations and is equal to the number of oscillations in \(2\pi\) units of time.

Substituting (1) and (4) in (2), we get the following equations in view of linear independence of the functions \(\cos \omega t\) and \(\sin \omega t\):

\[\Delta \varphi(x, y) + k^2 \varphi(x, y) = f(x, y)\]  \hspace{1cm} (4)

\[\varphi|_{\Gamma} = 0\]  \hspace{1cm} (5)

and

\[\Delta \psi(x, y) + k^2 \psi(x, y) = h(x, y)\]  \hspace{1cm} (6)

\[\psi|_{\Gamma} = 0\]  \hspace{1cm} (7)

where \(f = -\frac{\mu_1}{c^2}, \ h = -\frac{\mu_2}{c^2}, \ k = \frac{\omega}{c}\).

For the problem (4)-(5), we find the values of \(\varphi\) in the points of a network defined over the domain \(D\), using the finite differences schema and the Fast Fourier Transform (FFT) to calculate the coefficients of a trigonometric interpolation.
2. Problem formulation

Let us consider a Dirichlet problem for a non-homogeneous Helmholtz equation (4). To use the method of finite differences, it is necessary that the domain $D = [0, 1] \times [0, 1]$ to be replaced by a quadratic network with the step $h = 1/N$, and nodes $x_k = k/N, y_l = l/N$. Hence

$$D_N = \{(x, y) \mid 0 \leq k \leq N, 0 \leq l \leq N\}.$$

Let $\Gamma$ be the set of nodes of the form $(x_0, y_l), (x_l, y_0), (x_N, y_l), (x, y_0)$ and let function $\varphi(x)$ continuous on $D$ be the solution of (4) - (5). This is approximated by a reticular function $\varphi^h$. Defining $\varphi^h_{k,l} = \varphi(x_k, y_l)$, the Laplacian operator in node $(x_k, y_l)$ is of the form

$$(\Delta^h \varphi^h)_{k,l} = \frac{\varphi^h_{k+1,l} + \varphi^h_{k-1,l} + \varphi^h_{k,l+1} + \varphi^h_{k,l-1} - 4\varphi^h_{k,l}}{h^2}$$

and

$$\varphi^h_{k,l} = 0 \text{ on } \Gamma.$$  

(8)

Once the step $h$ was established we denote the reticular function $\varphi^h$ with $\varphi$ and problem (8) becomes

$$4\varphi_{k,l} - \varphi_{k-1,l} - \varphi_{k+1,l} - \varphi_{k,l-1} - \varphi_{k,l+1} = \frac{\mu \varphi_{k,l}}{h^2} + f(x_k, y_l)$$

$$(9)$$

$$\varphi(x_k, y_l) = 0 \text{ on } \partial D, k, l \in \{0, 1, \ldots, N\}.$$  

(10)

Introducing the notations

$$\varphi_l = \begin{bmatrix} \varphi_{1,l} \\ \varphi_{2,l} \\ \vdots \\ \varphi_{N-1,l} \end{bmatrix}, \quad f_l = \begin{bmatrix} f_{1,l} \\ f_{2,l} \\ \vdots \\ f_{N-1,l} \end{bmatrix},$$

we obtain the following set of equations

$$B\varphi_1 - \varphi_2 = h^2 f_1$$

$$- \varphi_{l-1} + B\varphi_l - \varphi_{l+1} = h^2 f_l, \quad l = 2, \ldots, N-2$$

(10)

$$- \varphi_{N-2} + B\varphi_{N-1} = h^2 f_{N-1}$$

(11)

The solutions of the eigenvalues problem

$$Bv^m(\omega) = \lambda_m(B)v^m(\omega)$$

is of the form

$$\lambda_m(B) = -2 \left[1 + \cos \left(\frac{(N-m)\pi}{N}\right) \right] + h^2 k^2 = 2$$

(12)

and

$$v^m_k = (-1)^k \frac{2}{N} \sin \frac{mk\pi}{N}, \quad m = 1, 2, \ldots, N-1$$

(13)

are the components of the vector $v^m$.

Since the eigenvalues of the matrix $B$ are distinct, the corresponding eigenvectors are linearly independent. The factor $\frac{2}{\sqrt{N}}$ has been chosen since the basis is orthonormal in the space $\mathbb{R}^{N-1}$. Now we verify this property of the vectors $\{v^m\}$, $m = 1, 2, \ldots, N-1$. In $\mathbb{R}^{N-1}$ space the product scalar of the two vectors is of the form

$$\langle v, w \rangle = \sum_{k=0}^{N-1} v_k w_k, \quad v_0 = v_N = 0$$

(14)

Then, for $m \neq r$, we have

$$\langle v^m, v^r \rangle = \sum_{k=0}^{N-1} (-1)^{m+r} \frac{mk\pi}{N} \sin \frac{rk\pi}{N} =$$

$$= \frac{(-1)^{m+r}}{N} \sum_{k=0}^{N-1} \cos \frac{(m-r)k\pi}{N} - \frac{(-1)^{m+r}}{N} \sum_{k=0}^{N-1} \cos \frac{(m+r)k\pi}{N}.$$
\[
\begin{align*}
\frac{1}{2} \left[ 1 - e^{\frac{il\pi}{N}} \right] + \frac{1}{2} \left[ 1 - e^{-\frac{il\pi}{N}} \right] &= \begin{cases} 0, & \text{if } l \text{ even} \\ 1, & \text{if } l \text{ odd} \end{cases} \\
\end{align*}
\]

Then, for \( m \neq r \), \((v^{(m)}, v^{(r)}) = 0 \) and for \( m = r \)
\[
(v^{(m)}, v^{(m)}) = \frac{1}{N} \sum_{k=0}^{N-1} \cos \frac{2\pi mk}{N} = 0,
\]
and (20) is Parseval’s formula. We observe that the relations (18) differ from the Fourier coefficients \( \beta \) by a constant. Using (18) and (24), we get for \( \hat{h}f_l \) that

\[
F_{m,l} = \frac{1}{N} \sqrt{\frac{2}{N}} \beta_m
\]

for each fixed \( l \).

Let us described the method for the estimation of \( \beta_l \) coefficients using FFT, [3], [5], [7]. In order to this, a network on \([0,1]\) with \(2N\) nodes was considered for each fixed \( l \) and \( f_{0j} = f_{k1} = \ldots = f_{2N-1,l} = 0 \). To halve the computations number one separates the components of \( f_l \) with even index from those with odd index for each \( l \). Let

\[
u_j = f_{2j} + if_{2j+1}, \quad j = 0, 1, \ldots, N - 1
\]

and define Direct Discret Fourier Transform (DDFT)

\[
u_k^j = \exp \left( i \frac{2\pi kj}{N} \right), \quad j = 0, 1, \ldots, N - 1
\]

and \( \nu_k^j \) can be expressed matrically as:

\[
U = T \cdot C
\]

where

\[
U = (u_0, u_1, \ldots, u_{N-1})^t, \quad C = (c_0, c_1, \ldots, c_{N-1})^t
\]

and \( t \) – the operation of transposition.
Since \(|\nu| = 1\) and \(\tilde{\nu} = e^{-\frac{2\pi}{N}} = 1/\nu\) one obtains 
\(T\tilde{T} = \tilde{T}T = N\tilde{I}_N\), where \(\tilde{T}\) is the conjugate 
matrix of \(T\). It follows from the form of \(T\) that it is a 
nonsingular matrix so that \(T^{-1} = \frac{1}{N}\tilde{T}\). Thus, using 
(27) one obtains 
\[c_j = \frac{1}{N} \sum_{k=0}^{N-1} v_k^{-j} u_{k,j}, \quad j \in \{0,1,\ldots, N-1\}\]  
(28)
which define the Inverse Discrete Fourier Transform 
and they can be written in a matrix form as 
\[C = \frac{1}{N} \tilde{T}U\]  
(29)
Considering that \(v_k^{-j} = v_k^{-j}\) and (29) we get 
\[C = \frac{1}{N} P\tilde{T}U\]  
(30)
where \(P\) is a permutation matrix.

**Theorem.** [8]. Let \(D_{2N}\) be a network on the interval \([0,1]\), \(\{f_k\}\), 
\(k \in \{0,1,\ldots, 2N-1\}\) be the reticular function, where \(N = 2^p\), \(p \in \mathbb{N}\) and \(u_k\) defined by (27). Then the 
coefficients of trigonometrically polynomial (23) are 
computed from the formulas 
\[\alpha_j - i\beta_j = \frac{1}{2}(c_j + \bar{c}_{N-j}) - \frac{i}{2}(c_j - \bar{c}_{N-j}) \exp(-i\pi j/N), \quad \beta_0 = \beta_N = 0; c_N = c_0\]  
\[\alpha_{N-j} - i\beta_{N-j} = \frac{1}{2}(c_{N-j} + \bar{c}_j) - \frac{i}{2}(c_{N-j} - \bar{c}_j) \exp(i\pi j/N), \quad j = 0,1,\ldots, N.\]  
(31)

In general, the complex products number 
from FFT algorithm has value \(N \log_2 N\). If \(u\) is not 
defined by (26), the number of the computations decreases if the matrix \(T\) of \(N \times N\) 
dimensions is replaced by a product of sparse matrices.

**3. Numerical examples**

I. Let’s define a quadratic network with 
\(N = 4, \quad h = 1/4, \quad k^2 = 1\) and a periodic function 
\(f(x,y) = (1 - 2\pi^2) \sin \pi x \sin \pi y\) with period \(T_s = 1.\)

In this case, the matrix \(B\) from (12) is of the form 
\[
\begin{bmatrix}
-3.94 & 1 & 0 \\
1 & -3.94 & 1 \\
0 & 1 & -3.94
\end{bmatrix}
\]
and the eigenvalues have the values 
\(\lambda_1 = -2.523, \quad \lambda_2 = -3.94, \quad \lambda_3 = -5.35\)  
(33)

Generally, \(\lambda_{N/2}\) is arithmetic mean of the others 
eigenvalues.

Let us now find the eigenvector components 
corresponding to eigenvalues \(\lambda_m\) with \(m = 1, 2, 3\). These are
\[
\begin{align*}
\nu_1^{(m)} &= \frac{-1}{\sqrt{2}} \sin \frac{\pi m}{4}, \\
\nu_2^{(m)} &= \frac{1}{\sqrt{2}} \sin \frac{2\pi m}{2}, \\
\nu_3^{(m)} &= \frac{-1}{\sqrt{2}} \sin \frac{3\pi m}{4}
\end{align*}
\]
(34)
and (22) will be 
\[
\begin{align*}
\lambda_m \Phi_{m,1} - \Phi_{m,2} &= F_{m,1} \\
-\Phi_{m,1} + \lambda_m \Phi_{m,2} - \Phi_{m,3} &= F_{m,2} \\
-\Phi_{m,2} + \lambda_m \Phi_{m,3} &= F_{m,3}
\end{align*}
\]
(35)
where we have successively \(m = 1, 2, 3\).

To obtain the solutions of (35), the Fourier 
coefficients of the vectors \(f_i, i \in \{1, 2, 3\}\) have to be 
computed. For each fixed \(l\) we use \(2N = 8\) nodes on 
\([0,1]\), \(x_j = j/2N, \quad j = 0,1,\ldots, 2N-1\) and in this aim we 
deﬁne the matrix 
\[
\begin{bmatrix}
f_0 & f_1 & \cdots & f_7\end{bmatrix}^T = \begin{bmatrix} 0 & f_{1,l} & \cdots & f_{3,l} & 0 & 0 & 0 \end{bmatrix}^T,
\]
(36)
and 
\[
U = \begin{bmatrix}
i \cdot f_{1,l} \\
f_{2,l} + i \cdot f_{3,l} \\
0 \\
0
\end{bmatrix}
\]
In our case, the distribution of \(f_{k,l} \quad k = 1, 2, 3\) is
Since $v_{k}^{-j} = v_{k}^{N-j}$, the permutation matrix is obtained as

$$ P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix} \quad \text{(37)} $$

and using (30) we obtain for $l = 1$

$$ C = \frac{1}{4} PTU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 & v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 & 0 \\ v_2 & v_3 & 0 & 0 \\ v_3 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -9.36i \\ -13.24 - 9.3i \\ 0 \end{bmatrix} $$

and

$$ C = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -9.36i \\ -13.24 - 9.3i \end{bmatrix} \Rightarrow $$. $C = \frac{1}{4} \begin{bmatrix} -3.3 - 4.7i \\ -2.33 + 0.97i \\ 3.3 \\ 2.34 - 5.65i \end{bmatrix}$

From (31) and (25) we find for $l = 1$: $\beta_0 = 0$, $\beta_1 = -6.62$, $\beta_2 = 0$, $\beta_3 = 0$ and then

$$ F_{1,1} = -1.17, \quad F_{2,1} = 0, \quad F_{3,1} = 0; $$

for $l = 2$:

$$ F_{1,2} = -1.654, \quad F_{2,2} = 0, \quad F_{3,2} = 0 \quad \text{(38)} $$

for $l = 3$:

$$ F_{1,3} = -1.17, \quad F_{2,2} = 0, \quad F_{3,2} = 0. $$

Now we can verify Parseval’s formula for $l = 1$:

$$ 3 \sum_{k=1}^{3} (h^2 f_{k,1})^2 = 3 \sum_{m=1}^{3} F_{m,1}^2 \quad \text{(39)} $$

Indeed,$$ \frac{1}{4^3} \left( 9.36^2 + 13.24^2 + 9.36^2 \right) = 1.17^2. $$

We will identify the solutions of the system (35) for $m \in \{1, 2, 3\}$ with $\lambda_m$ from (33). The values $F_{m,k}$ lead to only matrix equation, which corresponds to $\lambda_1 = -2.52$:

$$ \begin{bmatrix} -2.53 & -1 & 0 \\ -1 & -2.53 & -1 \\ 0 & -1 & -2.53 \end{bmatrix} \begin{bmatrix} \Phi_{1,1} \\ \Phi_{1,2} \\ \Phi_{1,3} \end{bmatrix} = \begin{bmatrix} 1.17 \\ -1.654 \\ 1.17 \end{bmatrix} $$

in accordance to the signs of eigenvectors. From (21) and (34), the approximate solution $\varphi$ and the exact solution, $\varphi_c = \sin(\pi x)\sin(\pi y)$ are presented in the tables 2

<table>
<thead>
<tr>
<th>$x_k$</th>
<th>$\frac{1}{4}$</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{3}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi$</td>
<td>$\varphi_c$</td>
<td>$\varphi$</td>
<td>$\varphi_c$</td>
</tr>
<tr>
<td>$l=1$</td>
<td>0.53</td>
<td>0.5</td>
<td>0.74</td>
</tr>
<tr>
<td>$l=2$</td>
<td>0.75</td>
<td>0.71</td>
<td>1.06</td>
</tr>
<tr>
<td>$l=3$</td>
<td>0.53</td>
<td>0.5</td>
<td>0.74</td>
</tr>
</tbody>
</table>

II. Now will be considered the same example, but with $N = 2^3$. The eigenvalues are: $\lambda_1 = -2.14$, $\lambda_2 = -2.6$, $\lambda_3 = -3.2$, $\lambda_4 = -3.9$, $\lambda_5 = -4.7$, $\lambda_6 = -5.4$, $\lambda_7 = -5.8$ and the numerical solution is presented in table 3.

<table>
<thead>
<tr>
<th>$\varphi_k (x_k)$</th>
<th>$l=1$</th>
<th>$l=2$</th>
<th>$l=3$</th>
<th>$l=4$</th>
<th>$l=5$</th>
<th>$l=6$</th>
<th>$l=7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>0.148</td>
<td>0.274</td>
<td>0.358</td>
<td>0.388</td>
<td>0.358</td>
<td>0.274</td>
<td>0.148</td>
</tr>
<tr>
<td>$\varphi_c$</td>
<td>0.274</td>
<td>0.506</td>
<td>0.661</td>
<td>0.716</td>
<td>0.661</td>
<td>0.506</td>
<td>0.274</td>
</tr>
</tbody>
</table>
III. Let us now consider the different values for constant \( k^2 \).

a. If \( k^2 = 10 \), the eigenvalues are:
\[
\lambda_1 = -1.96, \quad \lambda_2 = -3.8, \quad \lambda_3 = -5.23
\]
and the numerical solution

\[
\begin{array}{|c|c|c|c|}
\hline
l \quad x_k & 1/4 & 1/2 & 3/4 \\
\hline
l=1 & 0.55 & 0.786 & 0.55 \\
\hline
l=2 & 0.786 & 1.112 & 0.786 \\
\hline
l=3 & 0.55 & 0.786 & 0.55 \\
\hline
\end{array}
\]

b. If \( k^2 = 100 \), the eigenvalues are:
\[
\lambda_1 = 3.66, \quad \lambda_2 = 2.25, \quad \lambda_3 = 0.836
\]
and the numerical solution

\[
\begin{array}{|c|c|c|c|}
\hline
l \quad x_k & 1/4 & 1/2 & 3/4 \\
\hline
l=1 & 0.494 & 0.698 & 0.494 \\
\hline
l=2 & 0.698 & 0.988 & 0.698 \\
\hline
l=3 & 0.494 & 0.698 & 0.494 \\
\hline
\end{array}
\]

IV. If in the first case on replaced \( f \) with
\[
f(x, y) = xy(l - x)(y - 1) + 2(x + y - x^2 - y^2)
\]
we obtain the approximate solution \( k = 1 \):

\[
\begin{array}{|c|c|c|c|}
\hline
l \quad x_k & 1/4 & 1/2 & 3/4 \\
\hline
l=1 & -0.034 & -0.045 & -0.035 \\
\hline
l=2 & -0.048 & -0.064 & -0.048 \\
\hline
l=3 & -0.034 & -0.045 & -0.034 \\
\hline
\end{array}
\]

In this case, the exact solution is of the form:
\[
\phi_e = xy(1 - x)(y - 1)
\]
and in the point of network it has the values

\[
\begin{array}{|c|c|c|c|}
\hline
l \quad x_k & 1/4 & 1/2 & 3/4 \\
\hline
l=1 & -0.035 & -0.047 & -0.035 \\
\hline
l=2 & -0.047 & -0.063 & -0.047 \\
\hline
l=3 & -0.035 & -0.047 & -0.035 \\
\hline
\end{array}
\]

6. CONCLUSIONS

If we denote \( g(x) = \phi(x, y) \) for fixed \( l \) and \( x \in [0, 1] \), then it may be expanded in a trigonometric Fourier series at any point \( x \) of this interval:
\[
g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos 2\pi nx + b_n \sin 2\pi nx \right) \quad (40)
\]
where
\[
a_n = \frac{1}{l} \int_0^l g(x) \cos 2\pi nx dx, \quad b_n = \frac{1}{l} \int_0^l g(x) \sin 2\pi nx dx.
\]

Let us now consider \( P_N(x) \), the Fourier interpolation polynomial for \( g \) and we get
\[
g(x_j) = P_N(x_j), \quad x_j = \frac{i}{N}, \quad i = 0, 1, ..., N,
\]
\[
g(x_q) = g(x_N)
\]
where
\[
P_N(x) = A_0 + \sum_{k=1}^{N-1} \left( A_k \cos 2\pi kx + B_k \sin 2\pi kx \right).
\]

The increase of nodes number \( N \) of the network leads to a good approximation of the solution, which may be verified with discrete Parseval’s equality
\[
\frac{1}{4} A_0^2 + \frac{1}{2} \sum_{n=1}^{N-1} \left( A_n^2 + B_n^2 \right) = \frac{1}{2N} \sum_{i=0}^{2N-1} g^2(x_i)
\]

When \( N \) increases, the sum from the right part tends to \( \frac{1}{0} g^2(x)dx \). If the integral is bounded, the sum is bounded. Therefore, the sum of the left converge and \( A_n, B_n \) will be small. Thus, the trigonometric interpolation will converge for continuous function \( g \) or for function \( g \) with a finite number of points of discontinuity.

The numerical examples were obtained with the help of a program written in MathCAD. We show that the increasing of the \( k \) value leads to the increasing of the eigenvalues. Also, we find how depends the sign of the eigen-values of the \( k \). Besides, we observe that the error, \( e = \phi - \phi_e \), has negative value for positive eigenvalues and positive value for negative eigenvalues.

As before, it follows from the analysis of numerical results that the error tends to zero with increasing \( N \). The numerical results prove that the errors are smaller when the periodical force is a polynomial, than it is a periodical trigonometric force.

References