A Compact and Simple Form of Inertia Tensor

MOHAMMAD MEHDI FATEH
Department of Electrical and Robotic Engineering
Shahrood University of Technology, Shahrood, IRAN

Abstract: In this paper, formulation of inertia tensor for rigid bodies in a dynamic system is derived based on analytical method using skew symmetric matrixes. This formula describes the inertia tensor in a simple and compact form to simplify the related mathematical operations. The inertia tensor can be simply transformed in a desired coordinate frame by this formulation.

Key-Words: Inertia tensor, skew symmetric matrix, dynamic system, Hamel coefficients, robot dynamics.

1 Introduction

The principles and applications of tensors are usually considered in the linear algebra [1-4]. Robot dynamic is a very active area of tensor applications [5]. In robotics, the homogenous transformations, Denavit-Hartenberg representation Jacobian matrix of manipulator, dynamic equations, rigid motions and robot control are formulated using tensors. Matrix representation of dynamic equations can simplify the control design of a dynamic system and also its analysis. For example we can find many of such applications for a multi-link open-chain system [6].
Skew symmetric matrixes are very useful for deriving the dynamic equations of a manipulator. Some kind of transformation matrixes and skew symmetric matrixes are applied for a model-based control of a dynamic system [7]. It has been reported that skew symmetric matrixes are used for providing Hamel coefficients and deriving Lagrange equations for the n-dimensional rotation of a rigid body [8].
First of all, in this paper, we describe the features of skew symmetric matrix and rotation matrix. A novel formula for inertia matrix is then analytically derived based on kinetic energy of a rotational motion of a rigid body using the skew symmetric matrix. This formula which is presented in a simple and compact form can simplify the formulations which are related to the inertia matrix. After that, the transformations and descriptions of inertia matrix are formulated to transfer the inertia matrix from the center of mass coordinate frame to other frames which have significant applications in engineering. Finally, the derivative of the inertia matrix is also formulated for dynamic equations.

2 Skew Symmetric Matrix

S represents a 3×3 skew symmetric matrix [5] as follows

$$\mathbf{S}+\mathbf{S}^{\mathbf{T}}=\mathbf{0} \tag{1}$$

$$\mathbf{S(a)}=\begin{bmatrix} 0 & -a_x & a_y \\ a_x & 0 & -a_z \\ -a_y & a_z & 0 \end{bmatrix} \tag{2}$$

where $\mathbf{a}$ is a vector of the form $\mathbf{a}=\left[\mathrm{a}_x,\mathrm{a}_y,\mathrm{a}_z\right]^{\mathrm{T}}$.

$$\mathbf{S}(\alpha\mathbf{a}+\beta\mathbf{b})=\alpha\mathbf{S}(\mathbf{a})+\beta\mathbf{S}(\mathbf{b}) \tag{3}$$

where α and β are real constants, $\mathbf{a}$ and $\mathbf{b}$ are vectors.

$$\mathbf{a}\times\mathbf{b}=\mathrm{S}\mathbf{(a)b} \tag{4}$$

$$\dot{\mathbf{S}}\mathbf{(a)}=\mathbf{S}(\dot{\mathbf{a}})$$

$$\tag{5}$$

$$\int\mathbf{S(a)}\mathrm{dt}=\mathrm{S}(\int\mathbf{a}\mathrm{dt})$$

$$\mathbf{S(Ra)}=\mathbf{RS(a)R^T} \tag{6}$$

where $\mathbf{R}$ is a 3×3 rotation matrix which is an orthogonal matrix such as $\mathbf{RR^T}=\mathbf{I}$ where $\mathbf{I}$ is a unit matrix.

$$\dot{\mathbf{R}}=\mathbf{S}(\omega)\mathbf{R^T} \tag{7}$$

where $\boldsymbol{\omega} = \mathbf{a}\dot{\theta}$ that $\dot{\theta}$ is the angular velocity of the rotation matrix.

3 Deriving the Inertia Matrix

Assume a rigid body named b which rotates about center of its mass with no transferal motion. Attach a frame named C to the body such that its origin is coinciding on the center of mass. Position of a particle of the body in a reference frame named B which is fix and its origin is coinciding on the center of mass is calculated as

$$\mathbf{P_B} = \mathbf{R_B^C P_C} \tag{8}$$

where $\mathbf{P_c} = [x \quad y \quad z]^T$ is position of the particle in the frame C and $\mathbf{R_B^C}$ is a rotation matrix to describe the frame C in the frame B. Derivative of $\mathbf{P}_B$ can be computed as

$$\dot{\mathbf{P}}_\mathbf{B} = \dot{\mathbf{R}}_\mathbf{B}^\mathbf{C}\mathbf{P_C} \tag{9}$$

The kinetic energy K can be calculated as

$$K = \frac{1}{2}\int_b \dot{\mathbf{P}}_B^T \dot{\mathbf{P}}_B dm \tag{10}$$

Substituting $\dot{\mathbf{P}}_B$ from Equation (9) into Equation (10) leads to

$$K = \frac{1}{2}\int_b (\dot{\mathbf{R}}_\mathbf{B}^C \mathbf{P}_C)^\mathbf{T} (\dot{\mathbf{R}}_\mathbf{B}^C \mathbf{P}_C) dm \tag{11}$$

Use of Equation (7) as

$$\dot{\mathbf{R}}_\mathbf{B}^\mathbf{C} = \mathbf{S}(\boldsymbol{\omega}_\mathbf{B}^\mathbf{C})\mathbf{R}_\mathbf{B}^\mathbf{C} \tag{12}$$

Yields

$$K = \frac{1}{2}\int_b (\mathbf{S}(\boldsymbol{\omega}_\mathbf{B}^C)\mathbf{R}_\mathbf{B}^C \mathbf{P}_C)^\mathbf{T} (\mathbf{S}(\boldsymbol{\omega}_\mathbf{B}^C)\mathbf{R}_\mathbf{B}^C \mathbf{P}_C) dm \tag{13}$$

Then, using Equation (8) leads to

$$K = \frac{1}{2}\int_b (\mathbf{S}(\boldsymbol{\omega}_\mathbf{B}^C)\mathbf{P_B})^\mathbf{T} (\mathbf{S}(\boldsymbol{\omega}_\mathbf{B}^C)\mathbf{P_B}) dm \tag{14}$$

Next, by

$$\mathbf{S}(\boldsymbol{\omega}_\mathbf{B}^\mathbf{C})\mathbf{P_B} = -\mathbf{S}(\mathbf{P_B})\boldsymbol{\omega}_\mathbf{B}^\mathbf{C} \tag{15}$$

We have

$$\begin{aligned} K &= \frac{1}{2}\int_b (\mathbf{S}(\mathbf{P_B})\boldsymbol{\omega}_\mathbf{B}^C)^\mathbf{T} \mathbf{S}(\mathbf{P_B})\boldsymbol{\omega}_\mathbf{B}^C dm \\ &= \frac{1}{2}\int_b \boldsymbol{\omega}_\mathbf{B}^{C^\mathbf{T}} \mathbf{S}(\mathbf{P_B})^\mathbf{T} \mathbf{S}(\mathbf{P_B})\boldsymbol{\omega}_\mathbf{B}^C dm \end{aligned} \tag{16}$$

The angular velocity vector $\boldsymbol{\omega}_B^C$ is not dependent to m. Therefore,

$$K = \frac{1}{2}\boldsymbol{\omega}_B^{C^T}\left[\int_b \mathbf{S}(\mathbf{P_B})^\mathbf{T}\mathbf{S}(\mathbf{P_B})dm\right]\boldsymbol{\omega}_B^C \tag{17}$$

Rewriting the kinetic energy in the frame B as

$$K = \frac{1}{2}\boldsymbol{\omega}_B^{C^T}\mathbf{I}_B\boldsymbol{\omega}_B^C \tag{18}$$

Leads to

$$\mathbf{I_B} = \int_b \mathbf{S}(\mathbf{P_B})^\mathbf{T}\mathbf{S}(\mathbf{P_B})dm \tag{19}$$

I_B represents the inertia tensor of the rigid body b in the frame B as a simple and compact form. It can be rewrite in the frame C as

$$\mathbf{I}_C = \int_\mathbf{b} \mathbf{S}(\mathbf{P}_C)^\mathbf{T}\mathbf{S}(\mathbf{P}_C)dm \tag{20}$$

From Equation (2) and $\mathbf{P_c} = [x \quad y \quad z]^T$, then $\mathbf{S}(\mathbf{P}_C)$ becomes

$$\mathbf{S}(\mathbf{P}_C) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

(21)

Substituting $\mathbf{S}(\mathbf{P}_C)$ in Equation (20), leads to

$$\mathbf{I}_C = \begin{bmatrix} \int (y^2+z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (x^2+z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int (x^2+y^2)dm \end{bmatrix}$$

(22)

4 Transformation of Inertia Tensor

The inertia formula is then rewritten to transform the inertia tensor from one frame to other frame. Substituting $\mathbf{P}_B$ into Equation (19), yields

$$\mathbf{I_B} = \int_{\mathbf{b}} \mathbf{S}(\mathbf{R_B^C P_C})^{\mathbf{T}} \mathbf{S}(\mathbf{R_B^C P_C})\mathrm{dm} \tag{23}$$

Then, using Equation (6), leads to

$$\mathbf{I_B} = \int_{\mathbf{B}} \mathbf{R_B^C S(P_C)^T R_B^{C^T} R_B^C S(P_C) R_B^{C^T}}\mathrm{dm} \tag{24}$$

Since $\mathbf{R_B^{C^T} R_B^C} = \mathbf{I}$ and $\mathbf{R_B^C}$ is not dependent to dm, hence

$$\mathbf{I_B} = \mathbf{R_B^C}\left[\int_{\mathbf{B}} \mathbf{S(P_C)^T S(P_C)}\mathrm{dm}\right]\mathbf{R_B^{C^T}} \tag{25}$$

Then, substituting Equation (20) in Equation (25), leads to

$$\mathbf{I_B} = \mathbf{R_B^C I_C R_B^{C^T}} \tag{26}$$

Equation (26) is a transformation of inertia tensor from frame C to frame B. Then

$$\mathbf{R_B^{C^T} I_B R_B^C} = \mathbf{R_B^{C^T} R_B^C I_C R_B^{C^T} R_B^C} \tag{27}$$

Then $\mathbf{R_B^{C^T} R_B^C} = \mathbf{I}$ yields

$$\mathbf{I_C} = \mathbf{R_B^{C^T} I_B R_B^C} \tag{28}$$

Equation (28) is a transformation of inertia tensor from frame B to frame C.
In order to transform the inertia tensor from frame C to a desired frame named A, we continue as follows:
Position of a particle of the rigid body in the frame A named $\mathbf{P_A}$ is

$$\mathbf{P_A} = \mathbf{R_A^C P_C} + \mathbf{d_A^C} \tag{29}$$

where $\mathbf{d}_A^C$ is the origin of frame C described in frame A as $\mathbf{d}_\mathrm{A}^\mathrm{C} = [\mathrm{x_{CA}} \quad \mathrm{y_{CA}} \quad \mathrm{z_{CA}}]^\mathrm{T}$ and $\mathrm{R_A^C}$ is a rotation matrix to describe the frame C in the frame A. From Equation (20), we can write the inertia tensor in frame A as

$$\mathbf{I_A} = \int_{\mathbf{b}} \mathbf{S(P_A)^T S(P_A)}\mathrm{dm} \tag{30}$$

Substituting $\mathbf{P_A}$ into Equation (30), leads to

$$\mathbf{I_A} = \int_{\mathbf{b}} \mathbf{S}(\mathbf{R_A^C P_C} + \mathbf{d_A^C})^{\mathbf{T}} \mathbf{S}(\mathbf{R_A^C P_C} + \mathbf{d_A^C})\mathrm{dm} \tag{31}$$

That is

$$\begin{aligned}\mathbf{I_A} &= \int_{\mathbf{b}} \mathbf{S(R_A^C P_C)^T S(d_A^C)}\mathrm{dm} + \int_{\mathbf{b}} \mathbf{S(d_A^C)^T S(R_A^C P_C)}\mathrm{dm} \\ &+ \int_{\mathbf{b}} \mathbf{S(d_A^C)^T S(d_A^C)}\mathrm{dm} + \int_{\mathbf{b}} \mathbf{S(R_A^C P_C)^T S(R_A^C P_C)}\mathrm{dm}\end{aligned} \tag{32}$$

The first term in Equation (32) is simplified as

$$\int_{\mathbf{b}} \mathbf{S(R_A^C P_C)^T S(d_A^C)}\mathrm{dm} = \mathbf{R_A^C}\left[\mathbf{S}\left(\int_{\mathrm{b}} \mathbf{P_C}\mathrm{dm}\right)^\mathrm{T}\right]\mathbf{R_A^{C^T} S(d_A^C)} \tag{33}$$

Since $\int_\mathrm{b} \mathbf{P}_\mathrm{C}\mathrm{dm} = 0$, hence $\mathbf{S}(\int_\mathrm{b} \mathbf{P}_\mathrm{C}\mathrm{dm})^\mathrm{T} = 0$, then

$$\int_{\mathbf{b}} \mathbf{S(R_A^C P_C)^T S(d_A^C)}\, dm = 0 \tag{34}$$

The second term is the transpose of the first term. Therefore, it is zero as well. The third term is simplified as

$$\int_{\mathbf{b}} \mathbf{S(d_A^C)^T S(d_A^C)}\mathrm{dm} = \mathrm{m}\mathbf{S(d_A^C)^T S(d_A^C)} \tag{35}$$

The last term is simplified as

$$\begin{aligned}&\int_{\mathbf{b}} \mathbf{S(R_A^C P_C)^T S(R_A^C P_C)}\mathrm{dm} \\ &= \int_{\mathbf{b}} \mathbf{R_A^C S(P_C)^T R_A^{C^T} R_A^C S(P_C) R_A^{C^T}}\mathrm{dm} \\ &= \mathbf{R_A^C}\left[\int_{\mathbf{b}} \mathbf{S(P_C)^T R_A^{C^T} R_A^C S(P_C)}\mathrm{dm}\right]\mathbf{R_A^{C^T}} \\ &= \mathbf{R_A^C}\left[\int_{\mathbf{b}} \mathbf{S(P_C)^T S(P_C)}\mathrm{dm}\right]\mathbf{R_A^{C^T}} \\ &= \mathbf{R_A^C I_C R_A^{C^T}}\end{aligned} \tag{36}$$

As a result of substituting

$$\mathbf{I_A} = \mathbf{S}(\mathbf{d}_\mathbf{A}^\mathbf{C})^\mathbf{T}\mathbf{S}(\mathbf{d}_\mathbf{A}^\mathbf{C})\mathrm{m} + \mathbf{R}_\mathbf{A}^\mathbf{C}\mathbf{I_C}\mathbf{R}_\mathbf{A}^{\mathbf{C}^\mathbf{T}} \tag{37}$$

The inertia tensor in a expand form can be written as

$$\mathbf{I_A} = \begin{bmatrix} x_{CA}^2 + y_{CA}^2 & -y_{CA}z_{CA} & -x_{CA}z_{CA} \\ -y_{CA}z_{CA} & x_{CA}^2 + z_{CA}^2 & -x_{CA}y_{CA} \\ -x_{CA}z_{CA} & -x_{CA}y_{CA} & x_{CA}^2 + y_{CA}^2 \end{bmatrix} \mathrm{m}$$

$$+ \begin{bmatrix} \int (y^2 + z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (x^2 + z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int (x^2 + y^2)dm \end{bmatrix} \tag{38}$$

We can transform the inertia tensor from frame C to the desired frame A using Equation (37).
If frame A is parallel with frame C, then $\mathbf{R}_\mathbf{A}^\mathbf{C} = \mathbf{I}$ which yields $\mathbf{R}_\mathbf{A}^\mathbf{C}\mathbf{I_C}\mathbf{R}_\mathbf{A}^{\mathbf{C}^\mathbf{T}} = \mathbf{I_C}$. Therefore

$$\mathbf{I_A} = \mathbf{S}(\mathbf{d}_\mathbf{A}^\mathbf{C})^\mathbf{T}\mathbf{S}(\mathbf{d}_\mathbf{A}^\mathbf{C})\mathrm{m} + \mathbf{I_C} \tag{39}$$

5 Conclusion

The inertia tensor has been analytically formulated using skew symmetric matrixes. The inertia formula has been derived in a compact and simple form to simplify related mathematical operations. The transformation of inertia tensor from one frame to another frame can be simply written by use of the compact form of inertia tensor. The formula can be used in the derivative of kinetic energy to analyze and control of dynamic systems.

References:

[1] Joshi W. "*Matrices and Tensors in Physics*", 3rd Edition, John Wiley and Sons, INC., 1996.

[2] Smith M. S, "*Principles and Applications of Tensor Analysis*", Howard W. Sams & Co., Inc., Indianapolis; New York, 1963.

[3] Kirkland S., Psarrakos P.J., and Tsatsomeros M.J, "On the Location of the Spectrum of Hyper tournament Matrices", *Linear Algebra and Its Applications*, 2001, 323: pp. 37-49.

[4] Psarrakos P.J. and Tsatsomeros M.J., "The Perron Eigenspace of Nonnegative Almost Skew-Symmetric Matrices and Levinger's Transformation", *Linear Algebra and Its Applications*, 2003, 360:pp.43-57.

[5] Spong M.W. and Vidyasagar M., "*Robot Dynamics And Control*", John Wiley and Sons, INC, 1989.

[6] Perreira N. D., "Motions, efforts and actuations in constrained dynamic systems: a multi-link open-chain example", *Smart Mater. Struct.* **8,** 1999, pp. 476-489.

[7] Papadopoulos, E. and Chasparis, G., "Analysis and Model-based Control of Servomechanisms with Friction", *ASME J. Dynamic Systems, Measurement and Control*, Vol. 126, No. 4 , 2004, pp. 911-915.

[8] Hurtado J. E. and Sinclair A. J., "Hamel coefficients for the rotational motion of an *N*-dimensional rigid body", Proceedings: Mathematical, *Physical and Engineering Sciences*, Vol. 460, No. 2052, 2004, pp. 3613 - 3630.