Numerical Solution of One Dimensional Nonlinear Longitudinal Oscillations in a Class of Generalized Functions

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Abstract

In this paper a new numerical method for solution of initial value problem for the second order nonlinear wave equation which describes the standing vibrations of a finite continuous and nonlinear string. Special auxiliary problem has been suggested and using the advantages of the suggested auxiliary problem, a finite difference scheme is constructed for the numerical solution of the main problem.

Keywords: Longitudinal Oscillation. Numerical modeling. In a class of discontinuous function. Nonlinear wave propagation.

1 Introduction

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The problems associated with wave propagation and standing oscillation in nonlinear fields have been important for physics and engineering, and today it is still an actual topic. One of the nonlinear wave equations has been studied by Zabusky N.J in[2] for the first time and the exact solution has been achieved. In [1],[9] special methods for finding the exact solution of some nonlinear wave equations have been developed. However, it is not always possible to obtain the exact solutions of these problems. It is well known that, numerical solutions play an important role, where obtaining exact solutions falls short. For this reason in this paper an original numerical method for finding the numerical solution of the problem for the second order nonlinear partial wave equation given below

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} K\left(\frac{\partial u}{\partial x}\right), \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x) \tag{1.3}$$

is suggested. Here the functions $u_0(x)$ and $u_1(x)$ are given functions (may be discontinuous as well) and the function $K(s)$ satisfy the following conditions:

(i) $K(s)$ is continuous and differentiable,

(ii) $K(0) \geq 0$ for $s \geq 0$,

(iii) $K'(s)$ changes of sign.

It is obvious that the equation (1.1) degenerates twice, in deed since $\frac{\partial}{\partial x} K(\frac{\partial u}{\partial x}) = K'(\frac{\partial u}{\partial x})\frac{\partial^2 u}{\partial x^2}$ and $K'(s) = 0, K(0) = 0$ the equation (1.1) degenerates to first order differential equation. If $K(s) = s$ the equation(1.1) describes the little vibration of the string, but if $K(s) = \frac{s^3}{3} + s$ the equation (1.1) describes a larger vibration of the string. In addition, since the equation (1.1) is nonlinear in general, it is impossible to find the exact solution of the Cauchy problem of the equation (1.1).

As it is seen from the physical structure of the problem $K(\frac{\partial u}{\partial x})\frac{\partial u}{\partial x}$, it must be a continuous function, however the second order derivatives of the solution may not exist. Because the classical solution of the problem (1.1)-(1.3) does not exist, the weak solution of this problem is defined as

Definition 1. A function $u(x,t)$ which satisfy the conditions (1.2),(1.3) is called a weak solution of the problem (1.1)-(1.3), if the following integral relation

$$\int_D \left\{ \frac{\partial u}{\partial t}\frac{\partial f}{\partial t} - K\left(\frac{\partial u}{\partial x}\right)\frac{\partial f}{\partial x} \right\} dxdt$$

$$+\int_{-\infty}^{\infty} u_0(x)\frac{\partial f(0,t)}{\partial t}dx - \int_{-\infty}^{\infty} u_1(x)f(0,t)dx = 0 \tag{1.4}$$

holds for every test function $f(x,t) \in W_{1,1}^2(\overset{o}{R}{}_l^2)$ *and* $f(x,T) = 0$, *here,* $W_{1,1}^2(\overset{o}{R}{}_l^2)$ *is a Sobolev's space on* $\overset{o}{R}{}_l^2$.

In literature there are many familiar homogenous finite differences schemes in which the differentiability property of the solution is neglected [3],[4],[7],[8].

In order to find the numerical solution of the problem (1.1)-(1.3) in accordance to [5],[6] we introduce the following auxiliary problem .

2 Auxiliary problem

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The problem

$$\frac{\partial^2 v}{\partial t^2} = K\left(\frac{\partial^2 v}{\partial x^2}\right)\frac{\partial^2 v}{\partial x^2}, \tag{2.1}$$

$$v(x,0) = v_0(x), \tag{2.2}$$

$$\frac{\partial v(x,0)}{\partial t} = v_1(x). \tag{2.3}$$

is called an auxiliary problem. Here, the functions $v_0(x)$ and $v_1(x)$ are any continuous solutions of the equations

$$\frac{dv_0(x)}{dx} = u_0(x), \qquad \frac{dv_1(x)}{dx} = u_1(x). \tag{2.4}$$

Theorem 1. If $v_(x,t)$ *is the classical solution of the auxiliary problem (2.1)-(2.3), then the function defined by*

$$u(x,t) = \frac{\partial v(x,t)}{\partial x} \tag{2.5}$$

is weak solution of the main problem (1.1)-(1.3). With reference to (2.5) the auxiliary equation can be written as

$$\frac{\partial^2 v}{\partial t^2} = K\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}. \tag{2.6}$$

The auxiliary problem has the following advantages:

• The second order derivatives of $u(x,t)$with respect to x does not have to exist and it is not necessary to use this second order derivatives in the algorithm written to find the solution either.

• The $v(x,t)$ is absolutely continuous.

Both advantages of the auxiliary problem enables to construct the economical numerical algorithm for the problem (1.1)-(1.3).

3 Numerical algorithm in a class of discontinues functions

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In order to develop a numerical algorithm for the problem (2.1)-(2.3), at first, we cover the region R_l^2 by the grid as

$$\omega_{h,\tau} = \{(x_i, t_k) \mid x_i = ih, t_k = k\tau, i = 0, \pm 1, \pm 2, ..., k = 0, 1, 2, ...; h > 0, \tau > 0\}.$$

Here, h and τ are the steps of the grid $\omega_{h,\tau}$ with respect to x and t variables, respectively.

We approximate the problem (2.1),(2.3) to finite differences as follows

$$\frac{V_{i,k+1} - 2V_{i,k} + V_{i,k-1}}{\tau^2} = K\left(\frac{U_{i,k} - U_{i-1,k}}{h}\right)\left(\frac{U_{i,k} - U_{i-1,k}}{h}\right) \tag{3.1}$$

$$V_{i,0} = v_0(x_i), \tag{3.2}$$

$$\frac{V_{i,1} - V_{i,0}}{\tau} = u_1(x_i) \tag{3.3}$$

The following theorem holds;

Theorem 2. If $V_{i,k+1}$ is the numerical solutions of the auxiliary problem (2.1)-(2.3), then the relations defined by

$$\frac{V_{i,k+1} - V_{i-1,k+1}}{h} = U_{i,k+1} \tag{3.4}$$

is numerical solution of the problem

$$\frac{U_{i,k+1} - 2U_{i,k} + U_{i,k-1}}{\tau^2} = \frac{1}{h}\left[K\left(\frac{U_{i+1,k} - U_{i,k}}{h}\right) - K\left(\frac{U_{i,k} - U_{i-1,k}}{h}\right)\right] \quad (3.5)$$

$$U_{i,0} = u_0(x_i), \quad (3.6)$$

$$\frac{U_{i,1} - U_{i,0}}{\tau} = u_1(x_i). \quad (3.7)$$

The difference scheme (3.1) is the second order with respect to τ, however, the order of it can be made higher by applying, for example, the Runge-Kutta method.

As it can be seen from (3.1)-(3.3), the suggested algorithms are very effective and economic from a computational point of view.

4 Conclusion

In this study an original method for obtaining the numerical solution of the Cauchy problem for the second order nonlinear partial equation is suggested. We have introduced an auxiliary problem whose differentiability property of solution is one order higher than the differentiability property of the solution of the main problem. By this auxiliary problem an economical and effective differences scheme has been developed.

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