Numerical Solution and Simulation of Traffic Flow Problem in a Class of Discontinuous Funtions

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July 12, 2006

Abstract

In this paper, a new method for obtaining a numerical solution of the Cauchy problem for a first order partial differential equation which describes the traffic flow on highway is suggested. For this purpose, an auxiliary problem having some advantages over the main problem, but equivalent to it, is introduced and is studied some properties of the numerical solution. Some results of the comparison of the exact and numerical solutions have been illustrated.

Keywords: Traffic flow. Finite differences scheme in a class of discontinuous functions, Numerical extended solution.

1 Introduction

.As it is known the traffic flow problems are reduced to find the solution of first order nonlinear equation as

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(\rho(x,t))}{\partial x} = 0 \tag{1.1}$$

with following

$$\rho(x,0) = \rho_0(x) \tag{1.2}$$

initial condition [3], [4], [9], [10]. The problem (1.1),(1.2) is called as the main problem.

Here, $\rho(x,t)$ is a density of vehicle per unit length of the highway at any point x and t, $Q(\rho(x,t))$ is a flux function passing through any x section at unit time, $\rho_0(x)$ is an initial distribution vehicles on highway.

From physical point of view the function $Q(\rho(x,t))$ must be a concave function, that is, $Q''(\rho) \leq 0$ and the function $\rho_0(x)$ may be a piecewise continuous or continuous function with compact support and having negative and positive slops too. In [2], [7], [11], [12] proved that there is such number T_0 that the derivatives of the solution of the problem (1.1), (1.2) with respect to x and t becomes infinite for any $t \geq T_0$. Hence, the solution of the problem (1.1), (1.2) in a classical meaning doesn't exist. On the order hand the solution of the problem (1.1),(1.2) obtained by the method of characteristics is a closed form

$$\rho(x,t) = \rho_0(x - Q'(\rho)t),$$

but it is often impossible to obtain an explicit expression for the unknown function. The obtained functional relation is called as the alternative form of the problem (1.1),(1.2).

In that case the numerical methods are only way out to find the solution. In order to solve the problem (1.1),(1.2) by numerical methods higher differentiable property is required on solution, of which this property does not occur in solution. But formal approximation of the problem (1.1),(1.2) by finite differences reduce to faulty results from physical point of view.

In literature, there are such homogeneous finite differences schemes that, written ignoring the jump points which appear in the solution. In [1], it has been noted that classical finite difference schemes when applied to (1.1),(1.2) result in so-called *numerical viscosity*. Such a numerical viscosity has a negative effect on the solution. More precisely, it causes the numerical propagation rate of the wave to be larger than the actual physical rate. Furthermore, there exist some other numerical methods employing the method of characteristics, see [5],[11].

When we include the concept of a weak solution, a new problem arises that the location and time of the discontinuous points are unknown. It is obvious that a weak solution defined for nonlinear differential equations automatically fulfills the known jump condition. However this statement is not valid for the soft solution.

For this reason in this paper the numerical method for obtaining the weak solution, the problem (1.1),(1.2) is suggested.

According to [6], [7], [8], we introduce the following as the auxiliary problem.

2 The Auxiliary Problem and its Advantages

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By $A(\cdot)$ we denote the operator of differentiation with respect to x as $\frac{\partial}{\partial x}(\cdot)$. In this notation the equation (1.1) is written as

$$\frac{\partial \rho(x,t)}{\partial t} + AQ(\rho(x,t)) = 0. \tag{2.1}$$

Let A^{-1} is inverse operator of A. We consider the following equation

$$\frac{\partial A^{-1}\rho(x,t)}{\partial t} + Q(\rho(x,t)) = h(t), \tag{2.2}$$

here, $h(t) \in \ker A$. We introduce notation

$$A^{-1}\rho(x,t) = v(x,t) + h(t). \tag{2.3}$$

In this notation the equation (2.2) get the form

$$\frac{\partial v(x,t)}{\partial t} + Q(\rho(x,t)) = 0, \tag{2.4}$$

and its is called as the auxiliary equation.

The initial condition for (2.4) is

$$v(x,0) = v_0(x), \tag{2.5}$$

here $v_0(x)$ is any solution of $Av_0(x) = \rho_0(x)$.

The auxiliary problem has the following advantages: (i) the function $v(x,t)$ is absolutely continuous function; (ii) the equation (2.4) does not contain none derivatives of $\rho(x,t)$ with respect to x and t of which mentioned derivatives derivativies does not exist.

Theorem 1. If the function $v(x,t)$ is the soft solution of the problem (2.4), (2.5) then the function $\rho(x,t) = Av(x,t)$ is the weak solution of the main problem.

2.1 Numerical Algorithms and Computer Experiments

In this section, we intended to introduce the numerical method for the problem (1.1),(1.2), and investigate some properties of it. As emphasized in section 1, the solution of this problem has discontinuous points, whose location are unknown beforehand. The properties found in the exact solution do not permit the application of classical numerical methods to this problem directly. By using the advantages of the suggested auxiliary problem, a new numerical algorithm is proposed.

In order to construct the method, the domain of definition of the problem is covered by the following grid,

$$\omega_{h,\tau} = \{(x_i, t_k) \mid x_i = ih, t_k = k\tau, i = 0, \pm 1, \pm 2, ..., k = 0, 1, 2, ...; h > 0, \tau > 0\}$$

where, h and τ are steps of the grid for x and t variables, respectively.

Firstly, the problem of (2.4),(2.5) is approximated by the finite difference scheme at any point (i,k) of the grid $\omega_{h,\tau}$ as follows

$$V_{i,k+1} = V_{i,k} - \tau Q\left(\wp_{i,k}\right), \tag{2.6}$$

$$V_{i,0} = v_0(x_i). \tag{2.7}$$

Theorem 2. If the grid function $V_{i,k}$ is the numerical solution of the problem (2.4), (2.5) then the function $\wp_{i,k} = \left(V_{i,k} - V_{i-1,k}\right)/h$ is the solution of the following system of algebraic $\wp_{i,k+1} - \wp_{i,k} + \frac{\tau}{h}\left(Q(\wp_{i,k}) - Q(\wp_{i-1,k})\right) = 0$ equations.

Here, the grid functions $\wp_{i,k}$ and $V_{i,k}$ represent approximate values of the functions $\rho(x,t)$ and $v(x,t)$ at point (i,k) respectively.

Theorem 3. The expression $E_1(t_k) = h\sum_i U_{i,k}$ are independent of time.

Definition 1. The quantities $E_1(0)$ defined by $E_1(0) = h\sum_i U_{i,0}$ is called the critical values for the grid functions $V_{i,k}$.

Definition 2. The mesh functions defined by

$$V_{i,k}^{ext} = \begin{cases} V_{i,k}, & V_{i,k} < E_1(0) \\ E_1(0), & V_{i,k} \geq E_1(0), \end{cases} \tag{2.8}$$

are called the extended solutions of the problem (2.4),(2.5).

From Theorem 2, we have

$$\wp_{i,k}^{ext} = (V_{i,k}^{ext})_{\bar{x}}, \tag{2.9}$$

and these expression is called the extended numerical solution of the main problem.

Hence, the numerical solution of the main problem, which satisfies the energy integral defined by relations $\int_{-L}^{L} \rho(x,t)dx = const$ is obtained. As it can be seen from (2.4),(2.5), the suggested algorithms are very effective and economic from a computational point of view.

The suggested auxiliary equation (2.4) permits to write the higher order finite differences scheme with respect to τ. For example, using the familiar Runge-Kutta method to the equation (2.4), we can write a higher order finite difference scheme for the main problem with respect to τ.

In order to demonstrate the efficiency of the suggested numerical method and investigate the dynamical nature of traffic flow on highway we solve the algorithm (2.4),(2.5) with the following data. As the state function $Q(\rho)$ we have taken $Q(\rho) = \rho V(\rho) = v_{max}(1 - \frac{\rho}{\rho_{max}})\rho$. Here, v_{max} and ρ_{max} are the maximum values of speed of vehicles and density, respectively.

We have studied three cases: (i) $\rho_l = 0.3$ and $\rho_r = 0.1$; (ii) $\rho_l = 0.1$ and $\rho_r = 0.3$; (iii) the initial distribution of density has the following form

$$\rho_0(x) = \begin{cases} -\frac{\rho_{max}-\rho_{min}}{L^2}x^2 + \rho_{max}, & \mid x \mid \leq 1000 \\ 0, & \mid x \mid > 1000. \end{cases} \tag{2.10}$$

With these data some experiments have been performed on computer and when the exact solutions are compared with the ones obtained by the suggested numerical algorithm, it is obvious that both solutions are close to each other.

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