Investigating The Shock Wave of Traffic Flow On the Highway in A Class of Discontinuous Functions

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Abstract: In this paper, a method for obtaining an exact solution of the Cauchy problem for a first order partial differential equation which describes the traffic flow on highway is suggested. At first, a special auxiliary problem having some advantages over the main problem has been proposed and using the advantages of the auxiliary problem an original method for obtaining the weak solution of the main problem has been suggested.

Keywords: Traffic Flow. Auxiliary problem. Extended Solution. Differences scheme in a class of discontinuous functions.

1 Introduction

It is known that many problems of science and techniques have been reduced to find the solution of initial or initial-boundary problems for first order nonlinear partial differential equations [9],[10].

The traffic flow problem on highways is one of the above mentioned important problems. Using the kinematical theory of wave this problem has been presented for the first time in [3],[8]. The mathematical theory in detail has been investigated in [1],[5].

By $\rho(x,t)$ we denote the density of vehicles per unit length of highway, and $q(x,t)$ is the flux function of posing x section of the highway at per unit of time t, respectively. If we assume that there will be no vehicle joining to or leaving from the high way, the following balance equation

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx + q(x_1,t) - q(x_2,t) = 0 \qquad (1)$$

holds. As the first approach, in traffic flow problem the flux function $q(x,t)$ is expressed by the local density $\rho(x,t)$ as $q = Q(\rho) = \rho V(\rho)$.

If the functions $\rho(x,t)$ and $Q(\rho)$ are continuously differentiable then, the equation (1) is equivalent to

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(\rho(x,t))}{\partial x} = 0. \qquad (2)$$

We denote that in practical problems it is important to know the functional relation between Q and ρ. In order to investigate the dynamical structure of the solution of (2) we give the following inital condition

$$\rho(x,t) = \rho_0(x). \qquad (3)$$

The problem (2), (3) is called as the main problem. Here $\rho_0(x)$ is a given function and may be continuous with support function having both negative and positive slopes or a piecewise continuous function. From the physical point of view it must be $Q''(\rho) < 0$. That is $Q(\rho)$ is a concave function. The speed $Q'(\rho)$ of traffic flow wave is less than the speed of the vehicles in the traffic and the wave moving towards the opposite direction of the traffic flow informs the drivers about the happenings and situation of the traffic ahead. In order to investigate the dynamical nature of the traffic flow at first the equation (2) is investigated within following

$$\rho(x,0) = \begin{cases} \rho_l, & x < 0 \\ \rho_r, & x > 0 \end{cases} \qquad (4)$$

initial condition. Here, ρ_l and ρ_r are known constants. The problem (2), (4) is called the Riemann problem. There are two cases: (i) $\rho_l > \rho_r$, (ii) $\rho_l < \rho_r$.

Since $Q(\rho)$ is a concave function, according to the general theory when $\rho_l < \rho_r$ the solution of the problem (2), (4) has the points of discontinuity of which the location is not known beforehand and for this solution the entropy condition satisfies too. However, when $\rho_l > \rho_r$, in the solution the shock does not occur, but the first derivatives are first type discontinuous

[2], [4], [6]. It is clear that the classical solution for both cases does not exist. But, the weak solution of the problem is defined as

Definiton 1. The nonnegative and satisfying the initial condition (3) function $\rho(x,t)$ is called a weak solution of the problem (2),(3) if the following integral relation

$$\int_{D_T} \{\rho\varphi_t + Q(\rho)\varphi_x\}\, dxdt + \int_{-\infty}^{\infty} \varphi(x,0)\rho_0(x)dx = 0 \quad (5)$$

holds for every test functions $\varphi(x,t)$ and $\varphi(x,T) = 0$, and $D_T = \{-\infty < x < \infty, D \le t < T\}$.

The weak solution in the meaning of (5) of the problem (2),(3) consists of two parts; the continuous and shock part. The continuous differentiable part satisfies the equation (2) and the discontinuous part satisfies Rankino- Hugoniot condition [2],[4],[9],[10].

In this paper a new method for obtaining the weak solution of the problem (2),(3) is suggested.

2 Auxiliary Problem and Exact Solution

In order to find the weak solution of the problem (2),(3) the following auxiliary problem according to [6],[7]

$$\frac{\partial v(x,t)}{\partial t} + Q\left(\frac{\partial v(x,t)}{\partial x}\right) = 0, \quad (6)$$

$$v(x,0) = v_0(x) \quad (7)$$

is introduced. Here the function $v_0(x)$ is any solution of

$$\frac{dv_0(x)}{dx} = \rho_0(x). \quad (8)$$

Theorem 1. If $v(x,t)$ is the soft solution of the problem of (6),(7) then the function $\rho(x,t)$ obtained by

$$\rho(x,t) = \frac{\partial v(x,t)}{\partial x} \quad (9)$$

is a weak solution of the problem (2),(3) in the meaning of (5).

The solution of the problem (6),(7) obtained by characteristic method is

$$v(x,t) = \left[-Q\left(\frac{dv_0(\xi)}{d\xi}\right) + \left(\frac{dv_0}{d\xi}\right) Q'\left(\frac{dv_0}{d\xi}\right)\right] t + v_0(\xi), \quad (10)$$

here $\xi = x - Q'(\rho)t$, (see, [6],[7],[10]).

In order to show the dynamical structure of the traffic flow the flux function $Q(\rho)$ is defined as follows $Q(\rho) = v_m(1 - \frac{\rho}{\rho_m})\rho$, here v_m and ρ_m are the maximum values of the speed of vehicles and density, respectively.

With reference to [6] and [7] for $v(x,t)$ we have

$$v(x,t) = -\frac{v_m}{\rho_m}\rho^2(x,t)t + v_0(\xi). \quad (11)$$

By simple calculations and Theorem 1 for the function $\rho(x,t)$ we have the following

$$\rho(x,t) = \begin{cases} \rho_l, & \frac{x}{t} < v_m(1-\frac{2}{\rho_m})\rho_l, \\ -\frac{\rho_m}{v_m}\frac{x}{2t} + \frac{\rho_m}{2}, & v_m(1-\frac{2}{\rho_m})\rho_l < \frac{x}{t} < v_m(1-\frac{2}{\rho_m})\rho_r, \\ \rho_r, & \frac{x}{t} > v_m(1-\frac{2}{\rho_m})\rho_r \end{cases} \quad (12)$$

expression.

Now, we consider the case $\rho_l > \rho_r$, in this case the solution of the auxiliary problem has the following

$$v(x,t) = \begin{cases} v_-, & \xi < 0 \\ v_+, & \xi > 0 \end{cases} \quad (13)$$

form. Here,

$$v_- = -\frac{v_m}{\rho_m}\rho_l^2 t + \rho_l[x - Q'(\rho_l)]t = -\frac{v_m}{\rho_m}\rho_l^2 t + \rho_l\left[x - v_m\left(1 - \frac{2\rho_l}{\rho_m}\right)\right]t; \quad (14)$$

$$v_+ = -\frac{v_m}{\rho_m}\rho_r^2 t + \rho_r\left[x - v_m\left(1 - \frac{2\rho_r}{\rho_m}\right)\right]t. \quad (15)$$

It is clear that the shock of $v(x,t)$ from v_- to v_+ must be found through $v_- = v_+$. From this equation we have

$$\frac{x}{t} = v_m - \frac{v_m}{\rho_m}(\rho_r + \rho_l) \equiv S, \quad (16)$$

that this expression is the solution of $\frac{dx}{dt} = \frac{Q(\rho_r)-Q(\rho_l)}{\rho_r-\rho_l}$ which is known as Rankine-Hugoniot condition.

Taking Theorem 1 into account and (13) for the weak solution of the problem (2), (4) we get the following

$$\rho(x,t) = \begin{cases} \rho_l, & \frac{x}{t} < S \\ \rho_r, & \frac{x}{t} > S \end{cases} \quad (17)$$

expression.

Now, we assume that the function $\rho_0(x)$ is continuous and $supp\rho_0(x) \subset (-l, l)$. Since the equation (2) expresses the conservation law $E(t) = \int_{-l}^{l} \rho(x,t)dx = const$ and this integral relation must be conserved for both continuous and piecewise continuous functions.

Definition 3. The number $E(0)$ is called the critical number of the function of the problem (2), (3).

Definition 4. The function defined by the following expression

$$v_{ext}(x,t) = \begin{cases} -\frac{v_m}{\rho_m}\rho^2(x,t)t + v_0(\xi), & v(x,t) < E(0) \\ E(0), & v > E(0) \end{cases} \quad (18)$$

is called the extended solution of the auxiliary problem.

In accordance with the Theorem 1 for the weak solution of the main problem we have $\rho_{ext}(x,t) = \frac{\partial v_{ext}(x,t)}{\partial x}$.

3 Conclusion

In this paper a new method for obtaining the weak solution of the Cauchy problem for the first order nonlinear partial equation in a class of discontinuous function is suggested. By using this suggested method, it leads to obtaining and investigating a global solution for the Rieamann problem of the traffic flow on highways. Nevertheless, it gives ample opportunity to construct higher order numerical schemes for first order nonlinear partial differential equations.

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