Fuzzy Classification Analysis of Rules Usage on Probability Reasoning Test with Multiple Raw Rule Score

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Abstract: The purpose of this study is to analyze the rule usage of probability reasoning items by fuzzy partition with multiple rule score. There are four rules of solving strategies for probability reasoning test. The score of probability reasoning test depends on these four rules so that each subject has four scores as to rule usage. The scoring of the probability test differs from most related research. Moreover, the authors apply the fuzzy partition algorithm in the data analysis. Thanks to the fuzzy partition, subjects will be properly classified. Therefore, patterns of rule usage for each group display the cognitive structure of those who belong to that group. In the empirical study, it shows that patterns of rule usage vary with different grade of subjects, but not with gender.

Key-Words: fuzzy classification, probability reasoning, rule of problem-solving.

1 Background and Motivation

Events that require probabilistic reasoning are commonplace in daily life. For example, determining the likelihood of rain in weather forecasts and determining the likelihood of winning a lottery both pose common probabilistic problems. Psychological development researcher, like B. Inhelder and J. Piaget, acknowledged the importance of probabilistic reasoning in cognitive development [12]. Such reasoning is more evident among mature adolescents and adults; however, it can develop quite early in children [1].

People often make incorrect decisions regarding probabilistic events. Konold provided a set of questions, each called a “Weather Problem”, and found that subjects displayed misconceptions stemming from misinterpretation of the reasoning about uncertain events [10].

Garfield and Ahlgren claimed that the

understanding of ratio and proportion constitute a precondition to fine probabilistic reasoning [7] . In addition, they contended that the teaching of probabilistic concepts is very challenging, because many students have misconceptions regarding chance, randomness, and probability that would need to be weakened and replaced with corrected conceptions of the same probabilistic concepts.

The relationship between performance on probabilistic reasoning tasks and Piagetian conceptions of probability was investigated. It was found that probability learning tasks and Piagetian tasks assess similar cognitive skills and that performance on probability learning tasks reflects probabilistic reasoning [3]. DelMas and Bart employed an instructional activity on two groups of subjects with one group of subjects evaluating predictions based on intuitive understanding and the other group not performing the evaluation. Their study demonstrated that the evaluation group registered a significant increase in accurate predictions and the non-evaluation group did not [5].

Various important studies in cognitive development have focused on the topic of probabilistic reasoning. Piaget and Inhelder determined that adolescents in the period of formal operations are capable of manifesting probabilistic reasoning; whereas, individuals in lower cognitive periods are not [12]. As a result, they viewed probabilistic reasoning to be a characteristic of formal operations.

Fischbein provided the foundation of intuitive thinking as precursors to development of probability reasoning [6]. He claimed that there are two kinds of intuitions, which influence the development of probability reasoning. One is primary intuitions that are related to personal experiences and appear prior to instruction. The other is secondary intuitions that appear by way of the instructional process. As to the development of probability reasoning, Fischbein focused on the importance of intuitive thinking and advocated the research methodology of in-depth interviews. Children's probability intuitions on the expected value were also investigated. The results showed that intuition can support surprisingly precocious performance in young children and contribute to the biases evident in adult judgments and decisions [13].

Subjects do not use unique a rule when responding to probability reasoning problems. That is, subjects will change rules in the process of answering problems [15]. However, raw rule score could provide more information on patterns of rule usage but it is rarely discussed. In this study, the patterns of rule usage will be investigated by fuzzy partition algorithm.

Zadeh provided the concepts of fuzzy theory [16]. Fuzzy theory flourished well and brought contributions to data analysis. Fuzzy partition (or fuzzy c-means) is a soft classification method that also suits the data of social science [4] [9]. It is because human thinking is uncertain. Thus, membership of fuzzy set could explain this phenomenon appropriately [16].

In this study, raw rule score and patterns of rule usage will be discussed. The empirical data comes from elementary students in Taiwan and the fuzzy partition algorithm is applied. Thus, the patterns of rule usage will be investigated, as well as the development of rule usage based on age (grade) and gender.

2 Literature Review

As to the probability reasoning scoring, there are two kinds of scoring. One is raw score and the other is raw rule score. The meaning and clarification of these two scorings are explained as follows.

Raw score is applied in most of probability reasoning literature. This scoring simply focuses on the correct rule. The response conforming with the correct rule is coded as 1. Otherwise, the response is coded as 0. Namely, the score of items just depends on the answer key of correct rule. This raw score will lose information about incorrect rules usage due to the lack of distinguishing incorrect rules.

On the contrary, raw rule score depends on the information of all rules (correct and incorrect rules). The response is coded by the conformity of each rule. The multiple scores of each item can be considered as a vector. It can be found that the information of raw rule score covers that of raw score [11].

A concern of rule assessment methodology is the patterns of rule usage and developmental difference on rule usage [15]. As to this issue, raw rule score suit the goal of the data analysis.

Fuzzy partition is a method of clustering which allows one piece of data to belong to two or more clusters. It is frequently used in pattern recognition. As to the issue on patterns of rule usage, fuzzy partition can cluster subjects with similar patterns in the same cluster. In this study, Bezdek's algorithm is applied [4].

3 Method of Data Analysis

First of all, raw rule score depends on all rules (correct and incorrect ones). The response is coded by the conforming of each rule. For each rule, if the response conforms with the key answer of that rule, it is coded as 1. Otherwise, it is coded as 0. Therefore, each item results in multiple scores. The multiple scores of each item can be considered as a vector. There are totally four rules in probability reasoning test. For example, if the key answers of a certain item are (B A B E) based on these four rules, one subject's score on this item will be (1 0 1 0) if his response is B. Therefore, each subject has four total scores with respect to all rules. The data matrix of four total scores is the original data for fuzzy partition in this study.

Secondly, as to the fuzzy partition, suppose there are N subjects and each subject owns M variables. The data matrix is $X = (x_{nm})_{N \times M}$. The membership matrix $U = (u_{cn})_{C \times N}$ and the group center matrix $V = (v_{cm})_{C \times M}$ are unknown with group number C. The following objective function with optimization problem is applied.

$$J(U,V) = \sum_{n=1}^{N}\sum_{c=1}^{C}(u_{cn})^q d^2(c,n) \qquad (1)$$

where $d^2(c,n) = \sum_{m=1}^{M}(x_{nm} - v_{cm})^2$ and $1 \le q < \infty$. $J(U,V)$ is subject to $u_{cn} \in [0,1]$, $0 < \sum_{n=1}^{N} u_{cn} < N$ and $\sum_{c=1}^{C} u_{cn} = 1$. u_{cn} and v_{cm} are as follows:

$$u_{cn} = \frac{1}{\sum_{l=1}^{C}\left[\frac{d^2(c,n)}{d^2(l,n)}\right]^{\frac{1}{q-1}}} \qquad (2)$$

$$v_{cm} = \frac{\sum_{n=1}^{N}(u_{cn})^q (x_{nm})}{\sum_{n=1}^{N}(u_{cn})^q} \qquad (3)$$

u_{cn} and v_{cm} are acquired by iteration. The iteration does not stop until the two conditions (4) (5) are satisfied.

$$\left|u_{cn}^{(t+1)} - u_{cn}^{(t)}\right| < \delta \qquad (4)$$

$$\left|v_{cm}^{(t+1)} - v_{cm}^{(t)}\right| < \delta \qquad (5)$$

C is decided by partition validity. Partition entropy $H(U;C) = \frac{-1}{N}\sum_{n=1}^{N}\sum_{c=1}^{C} u_{cn} ln(u_{cn})$ and partition coefficient $F(U;C) = \frac{1}{N}\sum_{n=1}^{N}\sum_{c=1}^{C}(u_{cn})^q$ will be used to decide C [4]. For these two partition validity indices, the larger $F(U;C)$ is, the better partition number C will be. The smaller $H(U;C)$ is, the better partition number C will be.

4 Research Design

There are 959 valid subjects from 10 elementary schools in Taiwan. Of all the subjects, 336 subjects are fourth graders (9 years old), 325 subjects are fifth graders (10 years old) and 298 subjects are sixth graders (11 years old). The instrument is the probability reasoning test designed by authors.

Item 1 is an example to explain the test design. In Fig. 1, there are two sets containing black and white marbles. Set A contains two black marble and four white marbles. Similarly, set B contains one black marble and two white marbles. Let us represent item 1 as (2, 4) vs. (1, 2) that indicates that there were two black marbles and four white marbles in set A and 1 black marble and 2 white marbles in set B. The subject was asked to imagine picking one marble randomly from the two sets respectively. Subjects were requested to decide which set provides greater chance of picking a black marble, set A or set B or "equal." If these two sets have the same chance of picking a black marble, subjects must choose "equal."

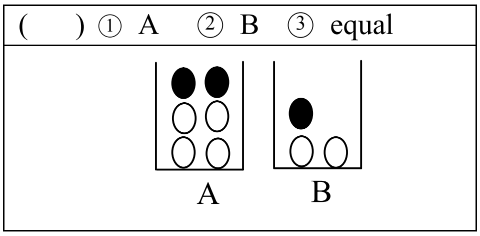

Fig. 1. Item 1 (2,4) vs. (1,2) in the test

It is indicated that there were three defective rules and one correct rule used by participants when solving marbles problems [2] [14]. Figs.2,3,4 present the flowchart of these three defective rules, and Fig. 5 present the flow chart of the correct rule. As Bart and Orton depicted these four rules, the following conventions are used [2] : (a) RB = number of black marbles on the right side (set B); (b) RW = number of white marbles on the right side (set B); (c) LB = number of black marbles on the left side (set A); and (d) LW = number of white marbles on the left side (set A).

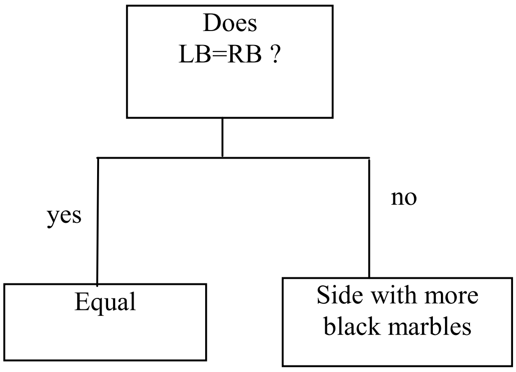

Fig. 2. Flowchart of Rule 1

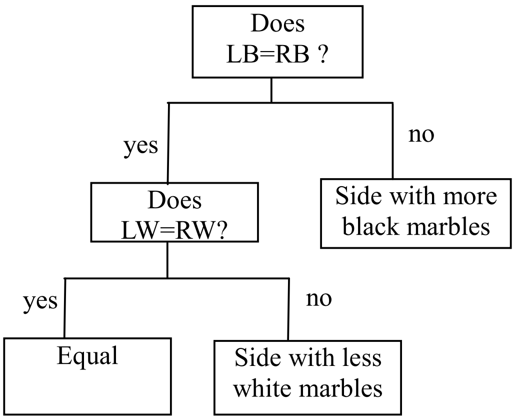

Fig. 3. Flowchart of Rule 2

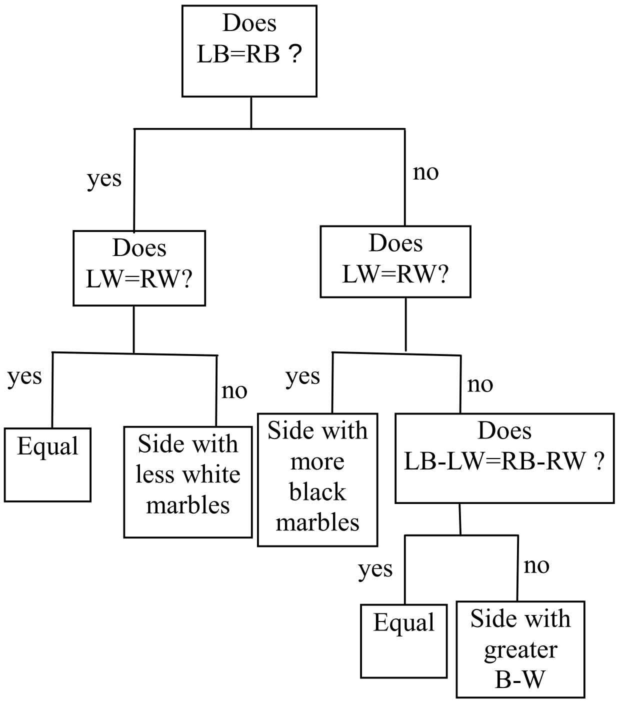

Fig. 4. Flowchart of Rule 3

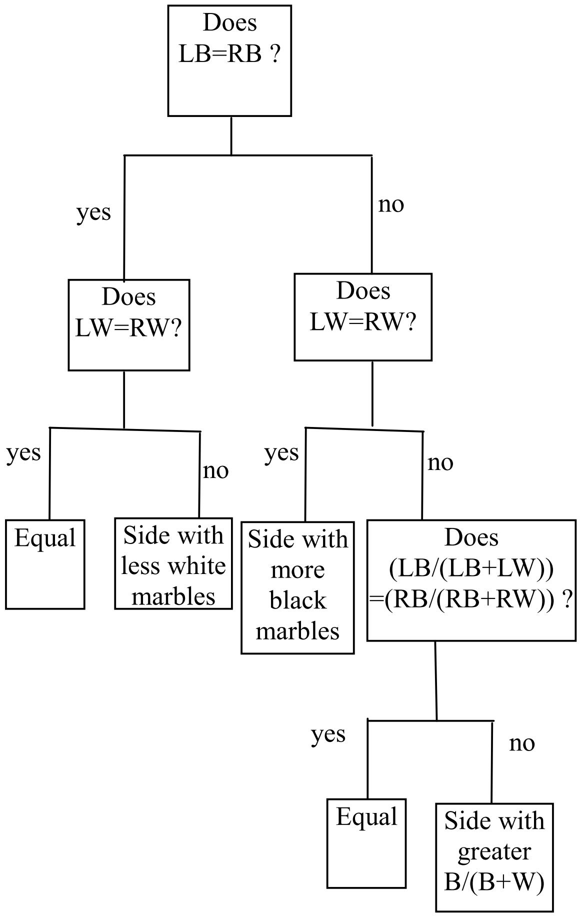

Fig. 5. Flowchart of Rule 4

Like the representation (a, b) *vs.* (c, d) used above, the compositions and answers of rules for 10 items in probability reasoning of this study are as Table 1 indicates. In Table 1, rule 4 is the correct rule.

Table 1. Items of Probability Reasoning Test

Item	Composition	Rules			
		1	2	3	4
1	(2, 4) vs.(1, 2)	A	A	B	E
2	(4, 2) vs.(6, 4)	B	B	E	A
3	(2, 4) vs.(1, 3)	A	A	E	A
4	(3, 7) vs.(3, 2)	E	B	B	B
5	(2, 3) vs.(2, 4)	E	A	A	A
6	(4, 6) vs.(2, 3)	A	A	B	E
7	(4, 2) vs.(4, 5)	E	A	A	A
8	(1, 3) vs.(2, 6)	B	B	A	E
9	(3, 1) vs.(4, 2)	B	B	E	A
10	(3, 6) vs.(2, 4)	A	A	B	E

5 Results

The raw rule score of subjects is coded and the possible highest score respective to each rule is 10. Fuzzy partition analysis is used to cluster subjects. Partition validity is shown in Table 2. Owing to the largest partition coefficient and the smallest partition entropy on group number 4, the clustering number 4 is selected.

As to this 4 group number, Table 3 is the vector of group center and its line chart is depicted as Fig. 6 shows. One can find that group I displays proficiency in rule 2; group II displays proficiency in rule 1; group III displays proficiency in rule 3 ; and group IV displays proficiency in rule 4. Therefore, subjects of group IV own the highest cognitive capacity while subjects of group II own the lowest cognitive capacity.

Table 2. Partition Validity

Group Number	Partition Coefficient	Partition Entropy
2	.991012	.026672
3	.994344	.018830
4	.999053	.002755
5	.989844	.017744
6	.991238	.015246

Table 3. Vector of Group Center

Group	Rules			
	1	2	3	4
I	7.2371	9.0999	2.6163	3.4316
II	10.000	6.9990	.0003	1.0000
III	.1329	3.0286	9.7257	3.0328
IV	1.0419	3.9752	3.0670	9.7328

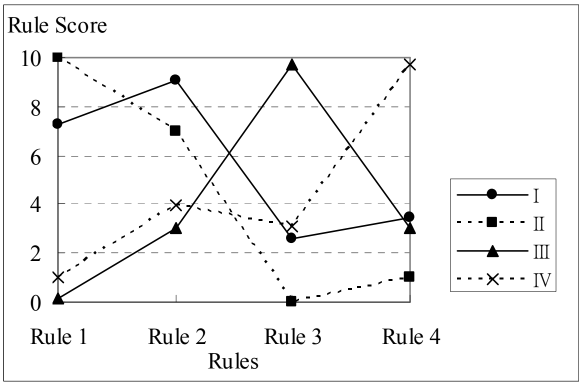

Fig. 6. Line Chart of Group Center

One of the goals of this study is to discuss the development of rule usage based on age (grade). Subjects are categorized into groups according to their maximum membership of the group. The frequencies of cross table based on grade and group is depicted in Table 4.

Table 4. Cross Table of Grade and Group

Grade	Group				Total
	I	II	III	IV	
4	203	113	7	13	336
5	79	65	156	25	325
6	15	4	77	202	298
Total	297	182	240	240	959

As to Table 4, the statistical analysis of chi-square is calculated and it shows Pearson Chi-square $\chi^2 = 706.611$ ($df = 6$, $p < .001$). Therefore, the frequencies for four groups within each grade are significantly different. Furthermore, subjects of grade 4 are located more on group I and group II. On the contrary, subjects of grade 5 are located more on group III and grade 6 are located more on group IV. The phenomenon indicates that patterns of rule usage vary with age (grade). Hence, there is cognitive difference of probability reasoning on age (grade).

Another goal of this study is to discuss the development of rule usage based on gender. The frequencies of cross table based on gender and group is depicted in Table 5.

Table 5. Cross Table of Gender and Group

Gender	Group				Total
	I	II	III	IV	
Male	151	89	120	121	481
Female	146	93	120	119	478
Total	297	182	240	240	959

As to Table 5, the statistical analysis of Chi-Square is calculated and it shows Pearson Chi-square $\chi^2 = .179$ ($df = 3$, $p \geq .05$). Therefore, the frequencies for four groups within each grade are not significantly different. It means that there is no relationship of rule usage and gender in probability reasoning.

6 Conclusions

The raw rule score is calculated and this score is seldom discussed in most rule assessment methodology. However, with the raw rule score, more information could explain the phenomenon of rule usage. Furthermore, fuzzy partition is used to analyze the patterns of rule usage so that development of rule usage is clarified. To sum up, the integrated method of raw rule score and fuzzy partition provide an alternative approach of rule assessment methodology.

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