An Approximate Solution for a Modified Van der Pol Oscillator

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Abstract: - In this paper, a Van der Pol oscillator containing a periodic oscillator is considered. A simple and effective iteration procedure to search the solution of the modified Van der Pol oscillator equation is proposed. This procedure is a powerful tool for determination of periodic solution of a non-linear equation of motion. The solutions obtained using the present iteration procedure are in good agreement with the numerical integration results obtained by a fourth order Runge-Kutta method, which shows the applicability of the procedure.

Key-Words: - Non-linear oscillations, modified Van der Pol oscillator, iteration procedure, periodic solution

1 Introduction

In the theory of non-linear oscillations, most attention is concentrated on the development of approximate analytical methods and their applications to specific models. In the last time, there has appeared an ever-increasing interest of scientists and engineers in the analytical techniques for non-linear problems.

Self-excited systems have a long history in the field of mechanics [1], [2]. A self-excited oscillator is a system which has some external source of energy upon which it can draw. Recently, self-excited systems have been proposed as fundamental tools for control and reduction of friction [3], [4]. In self-excited systems, the damping is a function of position and the most general picture can be described by the Lienard's differential equation

$$\ddot{x} + \mu f(x)\dot{x} + g(x) = 0 \quad (1)$$

Equations of this kind arise directly in various mechanical applications. One of the most studied equations within this class is the Van der Pol equation, which possess a unique limit cycle.

In this paper we will consider non-linear oscillations governed by the modified Van der Pol equation:

$$\frac{d^2x}{dt^2} + x + \mu(x^2 - 1)\frac{dx}{dt} + \Delta \sin x = 0 \quad (2)$$

where μ and Δ are positive parameters.

The classical Van der Pol equation is a typical example of a self-excited system. It has been studied extensively by many authors using various methods. Davis [5] and Urabe [6] obtained the steady-state solutions for various values of μ using numerical methods. Davis and Alfriend [7] and Chen et al [8] have obtained a perturbation solution for small values of μ using the two-variable expansion procedure with non-linear scales in the method of multiple scales. Bavinck and Grasman [9] is extending the method of matched asymptotic expansions obtained an approximation of the periodic solution connecting five local solutions. This method is suitable for large μ (e.g. $\mu \geq 0$). Another kind of methods, as described by Nayfeh [1], Stoker [10] and Mickens [11] is the perturbation methods including the method of multiple time scales, the Lindstedt-Poincare method and the Krylov-Bogoliubov-Mitropolski method: Burton [12] has analyzed the limit cycles for values of μ of order of magnitude unity by using a time transformation. For small μ to moderate μ (e.g. $\mu = 2$) the results obtained by this method are more accurate than those obtained by classical perturbation approaches.

Dadfar et al [13] have constructed a power series expansion in powers of ε for limit cycle of the Van der Pol equation up to $0(\mu^{24})$. Natsiavas [14] has examined the dynamics of piecewise linear oscillators with damping of the Van der Pol type. The limit cycle of a generalized Van der Pol equation in the form $\ddot{u} + u = \mu(1 - u^{2n})\dot{u}$ was investigated by Moremedi et al. [15].

The variational iteration method [16] does not depend on small parameters. A correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory. Several other generalizations of the Van der Pol equation have been considered in the literature: Nguyen [17] has examined the qualitative behavior of the non-linear oscillations, corresponding to the Van der Pol equation. They investigated the presence of local and global bifurcations and considered their physical significance.

In this paper we will use an iteration procedure to find the transient solution and the steady-state solution of equation (2). In this method, the problems are initially approximated with possible unknowns and the approximate solution obtained by the proposed method rapidly converges to its exact solution.

2 The iteration procedure [19], [20]

In this paper we will consider the following non-linear equation

$$\ddot{x}+\omega^2 x = f(\Omega t, x, \dot{x}) \quad (3)$$

where ω and Ω are positive constants, f is assumed to be non-linear function of both x and $\dot{x}$, periodic of Ωt, which may be expanded in a Fourier series, and $\dot{x}=\frac{dx}{dt}$.

According to Refs.[16], [18], we can construct the following iteration formula:

$$x_n(t) = x_{n-1}(t) + \int_0^t \lambda(\tau,t)\Big[x''_{n-1}(\tau)+\omega^2 x_{n-1}(\tau) - f(\Omega\tau, \tilde{x}_{n-1}(\tau), \tilde{x}'_{n-1}(\tau))\Big]d\tau \quad (4)$$

where $\lambda(\tau,t)$ is called a general Lagrange multiplier, which can be identified optimally via variational theory [18]; $\tilde{x}_{n-1}$ is considered as a restricted variation: $\delta\tilde{x}_{n-1} = \delta\tilde{x}'_{n-1} = 0$ where $' = \frac{d}{d\tau}$. The Lagrange multiplier must be a solution of the equation:

$$\begin{aligned} &\lambda''(\tau,\cdot)+\omega^2\lambda(\tau,\cdot)=0 \\ &\lambda(\tau,\cdot)\big|_{\tau=t}=0 \\ &1-\lambda'(\tau,\cdot)\big|_{\tau=t}=0 \end{aligned} \quad (5)$$

and can be readily identified

$$\lambda(\tau,t) = \frac{1}{\omega}\sin\omega(\tau-t) \quad (6)$$

As a result, we obtain the following iteration formula (named correction functional):

$$x_n(t) = x_{n-1}(\tau) + \frac{1}{\omega}\int_0^t \sin\omega(\tau-t)[x''_{n-1}(\tau) + \omega^2 x_{n-1}(\tau) - f(\Omega\tau, x_{n-1}(\tau), x'_{n-1}(\tau)]d\tau \quad (7)$$

or by taking into consideration the relation

$$\int_0^t \sin\omega(\tau-t)[x''_{n-1}(\tau)+\omega^2 x_{n-1}(\tau)]d\tau = -\omega x_{n-1}(t) + \omega x_{n-1}(0)\cos\omega t + x'_{n-1}(0)\sin\omega t \quad (8)$$

it follows that

$$x_n(t) = x_{n-1}(0)\cos\omega t + \frac{\dot{x}_{n-1}(0)}{\omega}\sin\omega t + \frac{1}{\omega}\int_0^t \sin\omega(\tau-t) f(\Omega\tau, x_{n-1}(\tau), x'_{n-1}(\tau))d\tau \quad (9)$$

Assume that $\omega \approx \Omega$ and let us denote (σ is detuning parameter)

$$\Omega^2-\omega^2=\varepsilon\sigma,\ 0<\varepsilon<<1 \quad (10)$$

The equation (3) may then be written as:

$$\ddot{x}+\Omega^2 x = F(\Omega t, x, \dot{x}) \quad (11)$$

where

$$F(\Omega t, x, \dot{x}) = \varepsilon\sigma x + f(\Omega t, x, \dot{x}) \quad (12)$$

The homogeneous solution of Eq.(11) is $x = A\cos(\Omega t+\varphi)$ where A and φ are constants.

We try the input of starting function as:

$$x_0 = A_0\cos(\Omega t + \varphi_0) \quad (13)$$

According to Eq.(9) we propose the following iteration formula for Eq.(11):

$$x_n(t) = A_n\cos(\Omega t+\varphi_n) + \frac{1}{\Omega}\int_0^t F(\Omega\tau, x_{n-1}(\tau), x'_{n-1}(\tau))d\tau \quad (14)$$

Expanding $F(\Omega\tau, x_{n-1}(\tau), x'_{n-1}(\tau))$ in a Fourier series, we have:

$$F(\Omega t, x_{n-1}(\tau), x'_{n-1}(\tau)) = \sum_{p=0}^{P} a_p^{n-1}(A_{n-1},\varphi_{n-1},\Omega,\sigma)\cos p\Omega\tau + \sum_{r=0}^{R} b_r^{n-1}(A_{n-1},\varphi_{n-1},\Omega,\sigma)\sin r\Omega\tau \quad (15)$$

and therefore, the approximation of n-th order (14) becomes:

$$\begin{aligned} x_n(t) = A_n\cos(\Omega t+\varphi_n) + \frac{1}{\Omega}\Bigg[&\frac{1}{2}a_1^{n-1}(t\sin\Omega t) + \frac{1}{2}b_1^{n-1}\left(t\cos\Omega t + \frac{1}{2\Omega}\sin\Omega t\right) + a_0^{n-1} + \\ &+\sum_{p=2}^{P}\frac{a_p^{n-1}(\cos\Omega t-\cos p\Omega t)}{(p^2-1)\Omega} + \sum_{r=2}^{R}\frac{b_r^{n-1}(r\sin\Omega t-\sin r\Omega t)}{(r^2-1)\Omega}\Bigg] \end{aligned} \quad (16)$$

The solution (16) is chosen such that it contains no secular terms, which requires that coefficients a_1^{n-1} and b_1^{n-1} disappear, i.e.:

$$a_1^{n-1}(A_{n-1},\varphi_{n-1},\Omega,\sigma)=0,\ b_1^{n-1}(A_{n-1},\varphi_{n-1},\Omega,\sigma)=0 \quad (17)$$

For real systems, the expansion of the function F usually contains only a small number of harmonics.

3 The solution of equation (2) with the iteration procedure

A modified Van der Pol oscillator has recently proposed to describe a self-excited body sliding on a periodic potential [3], [4]. This autonomous modified Van der Pol oscillator is described by the following equation:

$$M\frac{d^2x}{dt^2} + \Gamma(x^2-1)\frac{dx}{dt} + \frac{2\pi b}{\lambda}\sin\left(\frac{2\pi x}{\lambda}\right) + kx = 0 \quad (18)$$

where M is the mass, Γ is damping coefficient, b is the strength of the periodic potential, λ is its period and k is the stiffness constant. The external potential is the sum of the elastic contribution and sinusoidal term and it is defined as:

$$U(x,t) = \frac{1}{2}kx^2 - \frac{\lambda b}{2\pi}\cos\left(\frac{2\pi x}{\lambda}\right) \quad (19)$$

U(x,t) presents an absolute minimum point in x=0, and a series of periodic minima.
Making the changes:

$$x=\frac{2\pi x'}{\lambda},\ \tau=\Omega t,\ \Omega^2=\frac{k}{M},\ \Delta=\frac{4\pi^2 b}{M\Omega^2\lambda},\quad \dot{x}'=\frac{dx}{d\tau},\ \mu=\frac{\Gamma}{M\Omega} \tag{20}$$

we obtain:

$$\ddot{x}+x+\mu(x^2-1)\dot{x}+\Delta\sin x=0 \tag{21}$$

where the prime is omitted. Eq.(21) is very close to an Van der Pol forced oscillator where, in addition, a periodic term is introduced.

In Eq.(21) we have ω=1 and assuming $\Omega\approx 1$, this becomes:

$$\ddot{x}+\Omega^2 x=F(x,\dot{x});\quad F(x,\dot{x})=\mu\left[\sigma x+(1-x^2)\dot{x}\right]-\Delta\sin x,\quad \Omega^2-1=\mu\sigma \tag{22}$$

with the initial conditions

$$x(0)=A,\ \dot{x}(0)=0 \tag{23}$$

Assuming that the input of starting function is (13), we obtain in the condition (23):

$$x_0=A\cos\Omega t \tag{24}$$

and therefore

$$F(x_0,\dot{x}_0)=\mu\left[\sigma A\cos\Omega t-(1-A^2\cos^2\Omega t)A\Omega\sin\Omega t\right]-\Delta\sin(A\cos\Omega t) \tag{25}$$

The term $\sin(A\cos\Omega t)$ can be expanded in the power series:

$$\sin(A\cos\Omega t)=A\cos\Omega t-\frac{A^3\cos^3\Omega t}{3!}+\frac{A^5\cos^5\Omega t}{5!}-\frac{A^7\cos^7\Omega t}{7!}+\frac{A^9\cos^9\Omega t}{9!}+\ldots \tag{26}$$

We rewrite powers of $\cos\Omega t$ in (26) in terms of cosine of multipliers of Ωt with the aid of the identity [22]:

$$\cos^{2n+1}\Omega t=\frac{1}{4^n}\sum_{k=0}^{n}\binom{2n+1}{n-k}\cos(2k+1)\Omega t \tag{27}$$

where

$$\binom{n}{p}=\frac{n!}{p!(n-p)!}\ ;\ \binom{n}{0}=1\ ;\ k!=1\cdot 2\cdot 3\cdot\ldots\cdot k\ ;\ k\in\mathbf{N}$$

By using Eqs.(26) and (27), Eq.(25) may be expressed in the form:

$$F(x_0,\dot{x}_0)=\mu\left[\sigma A\cos\Omega t+A\Omega\left(\frac{A^2}{4}-1\right)\sin\Omega t+\frac{A^3\Omega}{4}\sin 3\Omega t\right]-\Delta A\cos\Omega t+\frac{\Delta A^3}{24}\left(3\cos\Omega t+\cos 3\Omega t\right)-$$
$$-\frac{\Delta A^5}{1920}(10\cos\Omega t+5\cos 3\Omega t+\cos 5\Omega t)+\frac{\Delta A^7}{322560}(35\cos\Omega t+21\cos 3\Omega t+7\cos 5\Omega t+\cos 7\Omega t)-\frac{\Delta A^9}{10321920}(126\cos\Omega t+84\cos 3\Omega t+36\cos 5\Omega t+9\cos 7\Omega t+\cos 9\Omega t)+\ldots$$

or

$$\begin{aligned}F(x_0,\dot{x}_0)&=A\Omega\left(\frac{A^4}{4}-1\right)\sin\Omega t+A\cos\Omega t\Big(\mu\sigma-\Delta+\frac{\Delta A^2}{8}-\frac{\Delta A^4}{192}+\frac{\Delta A^6}{9216}-\frac{\Delta A^8}{737280}+\ldots\Big)+\frac{\mu\Omega A^3}{4}\sin 3\Omega t+\\&+\Delta A^3\cos 3\Omega t\left(\frac{1}{24}-\frac{A^2}{384}+\frac{A^4}{15360}-\frac{A^6}{1105920}+\ldots\right)-\\&-\Delta A^5\cos 5\Omega t+\left(\frac{1}{1920}-\frac{A^2}{46080}+\frac{A^4}{2580480}+\ldots\right)+\\&+BA^7\cos 7\Omega t\left(\frac{1}{322560}-\frac{A^2}{10321920}+\ldots\right)-\\&-BA^9\left(\frac{1}{92897280}+\ldots\right)\cos 9\Omega t+\ldots\end{aligned} \tag{28}$$

From the conditions (17) we obtain:

$$A=2;\ \mu\sigma=\Delta\left(1-\frac{A^2}{8}+\frac{A^4}{192}+\frac{A^6}{9216}-\frac{A^8}{737280}+\ldots\right)=\frac{1661}{2880}\Delta \tag{29}$$

and from eq.(22_3), we have the square of frequency:

$$\Omega^2=1+\frac{1661}{2880}\Delta \tag{30}$$

Substituting Eq.(29_1) into Eq.(28) we obtain:

$$F(x_0,\dot{x}_0)=2\mu\Omega\sin 3\Omega t+\frac{557}{2160}\Delta\cos 3\Omega t-\frac{71}{5040}\Delta\cos 5\Omega t+\frac{7}{20160}\Delta\cos 7\Omega t-\frac{1}{181440}\Delta\cos 9\Omega t \tag{31}$$

The solution (16) for n=1 is written in the form:

$$\begin{aligned}x_1(t)&=A\cos\Omega t+\frac{\mu}{4\Omega}(3\sin\Omega t-\sin 3\Omega t)+\\&+\frac{557}{17280}\frac{\Delta}{\Omega^2}(\cos\Omega t-\cos 3\Omega t)-\\&-\frac{71}{120960}\frac{\Delta}{\Omega^2}(\cos\Omega t-\cos 5\Omega t)+\\&+\frac{7}{967680}\frac{\Delta}{\Omega^2}(\cos\Omega t-\cos 7\Omega t)-\\&-\frac{1}{14515200}\frac{\Delta}{\Omega^2}(\cos\Omega t-\cos 9\Omega t)+\ldots\end{aligned}$$

or

$$\begin{aligned}x_1(t)&=\left(2+\frac{459464}{14515200}\frac{\Delta}{\Omega^2}\right)\cos\Omega t+\frac{\mu}{4\Omega}(3\sin\Omega t-\sin 3\Omega t)-\\&-\frac{557}{17280}\frac{\Delta}{\Omega^2}\cos 3\Omega t+\frac{71}{120960}\frac{\Delta}{\Omega^2}\cos 5\Omega t-\\&-\frac{7}{967680}\frac{\Delta}{\Omega^2}\cos 7\Omega t+\frac{1}{14515200}\frac{\Delta}{\Omega^2}\cos 9\Omega t+\ldots\end{aligned} \tag{32}$$

where Ω is given by Eq.(30).

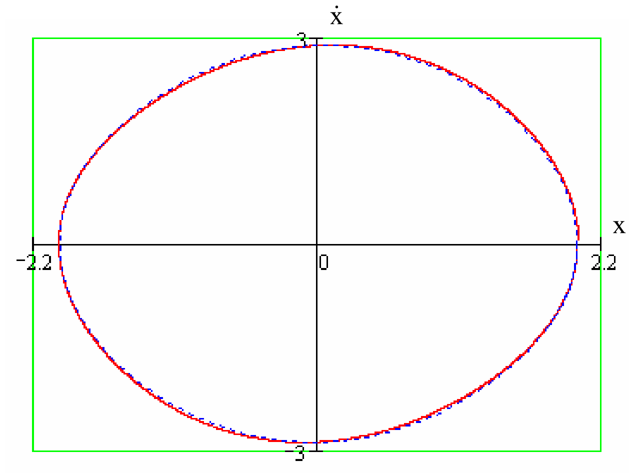

Fig.1: Phase space for Van der Pol modified oscillator, μ=0.1, Δ=1.5
——— numerical simulation
– – – – present method

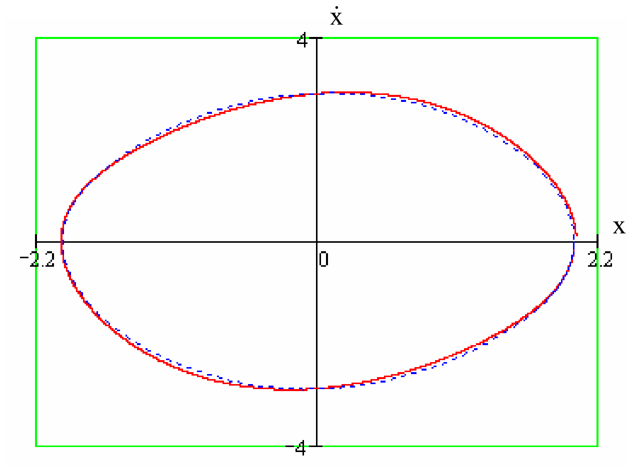

Fig.2: Phase space for Van der Pol modified oscillator, μ=0.2, Δ=1.5
——— numerical simulation
– – – – present method

Fig.1 and fig.2 show the comparison between the present phase space obtained from formula (32) and the numerical integration results obtained by using a fourth order Runge-Kutta method for μ=0.1 and μ=0.2, respectively. It can be seen that the solution obtained by the present method is nearly identical with that given by the numerical method.

One can conclude that adopting present procedure to analyze the solution of the modified Van der Pol equation, a satisfactory result can be obtained for small values of parameter μ

4 Conclusion

In this paper a new kind of analytical technique for modified Van der Pol equation is presented. The problem is initially approximated with unknown constants, which can be further determined. This procedure is effective and accurate for non-linear problems with approximations converging rapidly to accurate solution.

References:

[1] Nayfeh A.H., Mook D.T., *Nonlinear oscillations*, John Willey and Sons, N.Y., 1979

[2] den Hartog J.P., *Mechanical* Vibrations, Dover Publ. N.Y., 1984

[3] D'Acunto M, A simple model for low friction systems, *In: Bhushan B. (Ed), NATO ASI* Series, vol.311

[4] D'Acunto M, Determination of limit cycles for a modified Van der Pol oscillator, *Mechanics Research Communications*, 33, 2006, pp.93-98

[5] Davis N.T., *Introduction to nonlinear differential and integral* equation, Dover N.Y., 1962

[6] Urabe M, Numerical study of periodic solutions of the Van der Pol equation, *Nonlinear Differential Equations and Nonlinear Mechanics*, Academic Press, N.Y., 1963

[7] Davis R.T., Alfriend K.T., Solution to Van er Pol equation using a perturbation method, *International Journal of Nonlinear Mechanics*, 2, 1967, pp.153-162

[8] Chen S.H., Cheung Y.K., Lau S.L., On perturbation procedure for limit cycle analysis, *International Journal of Nonlinear Mechanics*, 1, 1993, pp.237-250

[9] Bavinck H., Grasman J., The method of matched asymptotic expansions for the periodic solution of the Van der Pol equation, *International Journal of Nonlinear Mechanics*, 9, 1974, pp.421-434

[10] Stoker J.J., *Nonlinear vibrations in mechanical and electrical systems*, Interscience Publishers N.Y., 1950

[11] Mickens R.E., Perturbation procedure for the Van der Pol oscillator based on Hopf bifurcation theorem, *Journal of Sound and Vibration*, 127, 1988, pp.187-193

[12] Burton T.D., Non-linear oscillator limit cycle analysis using a time transformation approach, *International Journal of Nonlinear Mechanics*, 17, 1982, pp.7-19

[13] Dadfar M.B., Geer J., Anderson C.M., Perturbation analysis of a limit cycle of the free Van der Pol equation, *SIAM Journal of Applied Mathematics*, 44, 1984, pp.881-895

[14] Natsiavas S., Dynamics of piecewise linear oscillators with Van der Pol type damping,

International Journal of Nonlinear Mechanics, 26, 1991, pp.349-366

[15] Mormedi G.M., Mason D.P., Gorringe V.M., On the limit cycle of a generalized Van der Pol equation, *International Journal of Nonlinear Mechanics*, 2, 1993, pp.237-250

[16] He J.H., Variational iteration method-a kind of non-linear analytical technique: some examples, *International Journal of Nonlinear Mechanics*, 34, 1999, pp.699-708

[17] Nguyen V.D., Some properties on the generalized Van der Pol equation, *Journal of Technical Physiscs*, 17, 1976, pp.183-190

[18] Marinca V., An approximate solution for one-dimensional weakly nonlinear oscillations, *International Journal of Nonlinear Science and Numerical Simulation*, 3, 2002, pp.107-120

[19] Marinca V., Herişanu N., A modified iteration perturbation method for some non-linear oscillation problems, *Acta Mechanica*, 184, 231-242, 2006

[20] Marinca V., Herişanu N., An iteration procedure for the analysis of non-linear one-dimensional oscillation, *Meccanica*, (to appear)

[21] Marinca V., Herişanu N., An iteration procedure with application to Van der Pol oscillator, (to appear)

[22] Gradshteyn I.S., Ryzhik I.M., *Tables of integrals, series and products*, Academic Press, N.Y., 1980

[23] Marinca V., Herişanu N., Application of an iteration procedure in solution of the Van der Pol equation, *Mechanics Research Comm.*, (to appear)

[24] Doedel E.J., Freire E., Gamero E., Rodriguez-Luis A.J., An analytical and numerical study of a modified Van der Pol oscillator, *J. of Sound and Vibration,* 256(4), 755-771, 2002

[25] Abd El-Latif G.M., Parametric excitation for forcing van der Pol oscillator, *Applied Mathematics and Computation,* 147, 255–265, 2004

[26] Lim C.W., Lai S.K., Accurate higher-order analytical approximate solutions to nonconservative nonlinear oscillators and application to van der Pol damped oscillators, *International Journal of Mechanical Sciences*, 48, 483–492, 2006

[27] Padin M.S., Robbio F.I., Moiola J.L., Chen G., On limit cycle approximations in the van der Pol oscillator *Chaos, Solitons and Fractals*, 23, 207–220, 2005

[28] Fotsin H., Bowong S., Adaptive control and synchronization of chaotic systems consisting of Van der Pol oscillators coupled to linear oscillators, *Chaos, Solitons and Fractals* 27, 822–835, 2006