Gauss-Legendre Quadrature Formula in Runge-Kutta Method with Modified Model of Newton Cooling Law

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Abstract: In this paper we introduce, the so called 'Open Formula', two points formula, three points formula, four points formula, five points formula and six points formula of the Runge-Kutta method to solve the initial value problem of the ordinary differential equation. These formulas use the points and weights from the Gauss-Legendre Quadrature formulas for finding the value of the definite integral. We will use these formulas to compute the numerical solution of the initial value problem of the ordinary differential equations from the models that are modified from The Newton Cooling Law.

Keywords: Gauss-Legendre, Runge-Kutta, Open, Quadrature Fehlberg

1 Introduction

The initial value problem of the ordinary differential equation is of the form

(1) $y'(x) = f(x, y), \quad x \in [a, b]$

with the initial condition

(2) $y(a) = c$.

The partition P over the closed interval [a,b] is the finite sequence of real numbers, $x_0, x_1, ..., x_n$ where $x_i = a + ih$ and $h = (b-a)/n$. The k-points Runge-Kutta formula, the explicit single step method, to find the value of function at the above points is of the form

(3) $y_{m+1} = y_m + \sum_{i=1}^{k} a_i K_i$

where $K_1 = f(x_m, y_m)$,

$K_2 = f(x_m + \alpha_1 h, y_m + \beta_1 h K_1)$,...

$K_k = f(x_m + \alpha_{k-1} h, y_m + h\sum_{j=1}^{k-1} \beta_{n_j} K_j)$

where $n_1 = 1 + (k-1)(k-2)/2$ and $n_{k-1} = k(k-1)/2$.

The most frequently used are the four points formula of Runge-Kutta-Fehlberg formula.

2 Formulation

The idea of our new formulas is to use the points $(x_m + \alpha_i h)$'s in our formula from the points in the Gauss-Legendre formula, i.e. the roots of the Legendre orthogonal polynomial, and the weights K_i's in our formula from the weights in the Gauss-Legendre Quadrature formula. Thus we have the following five formulas which we shall call them "the open formula".

Two Point Formula

The two points are $x_m + \alpha_1 h$, and $x_m + \alpha_2 h$ with $\alpha_1 = 0.5(1 - 1/\sqrt{3})$ and $\alpha_2 = 0.5(1 + 1/\sqrt{3})$. The two weights are $A_1 = A_2 = 0.5$. Thus our two points formula is of the form

(4) $$y_{m+1} = y_m + \frac{h}{2}(K_1 + K_2)$$

where

$$K_1 = f(x_m + \alpha_1 h, y_m + \alpha_1 h f(x_m, y_m))$$

$$K_2 = f(x_m + \alpha_2 h, y_m + \alpha_2 h K_1).$$

Three Point Formula

The three points formula is of the form

(5) $$y_{m+1} = y_m + \frac{h}{18}(5K_1 + 8K_2 + 5K_3)$$

where

$$K_1 = f(x_m + \frac{496}{4401}h, y_m + \frac{496}{4401}hf(x_m, y_m))$$

$$K_2 = f(x_m + \frac{h}{2}, y_m + \frac{h}{2}K_1)$$

$$K_3 = f(x_m + \frac{496}{559}h, y_m - \frac{74}{433}hK_1 + \frac{2491}{2354}hK_2).$$

Four Point Formula

The four points formula is of the form

(6) $$y_{m+1} = y_m + h\sum_{i=1}^{4} a_i K_i$$

where $a_1 = a_3 = \frac{18 - \sqrt{30}}{72}$

$a_2 = a_4 = \frac{18 + \sqrt{30}}{72}$

$$K_1 = f(x_m + \frac{842}{12127}h, y_m + \frac{842}{12127}hf(x_m, y_m))$$

$$K_2 = f(x_m + \frac{695}{2106}h, y_m + \frac{695}{2106}hK_1)$$

$$K_3 = f(x_m + \frac{739}{1103}h, y_m - \frac{362}{1175}hK_1 + \frac{803}{821}hK_2)$$

$$K_4 = f(x_m + \frac{1032}{1109}h, y_m + \frac{1379}{1602}hK_1 - \frac{1249}{1742}hK_2 + \frac{1664}{2115}hK_3).$$

Five Point Formula

The five points formula is of the form

(7) $$y_{m+1} = y_m + h\sum_{i=1}^{5} a_i K_i$$

where $a_1 = \frac{322 - 13\sqrt{70}}{1800}$

$a_2 = \frac{322 + 13\sqrt{70}}{1800}$

$a_3 = \frac{512}{1800}$

$a_4 = a_2$, $a_5 = a_1$ and

$$K_1 = f(x_m + (\frac{1}{2} - \frac{1}{42}\sqrt{245 + 14\sqrt{70}})h, y_m + (\frac{1}{2} - \frac{1}{42}\sqrt{245 + 14\sqrt{70}})hf(x_m, y_m))$$

$$K_2 = f(x_m + (\frac{1}{2} - \frac{1}{4}\sqrt{245 - 14\sqrt{70}})h, y_m + (\frac{1}{2} - \frac{1}{4}\sqrt{245 - 14\sqrt{70}})hK_1))$$

$$K_3 = f(x_m + \frac{h}{2}, y_m - \frac{1163}{3926}hK_1 + \frac{676}{849}hK_2)$$

$$K_4 = f(x_m + (\frac{1}{2} + \frac{1}{4}\sqrt{245 - 14\sqrt{70}}h), y_m + \frac{1578}{1513}hK_1 - \frac{2989}{2531}hK_2 + \frac{265}{293}hK_3)$$

$$K_5 = f(x_m + (\frac{1}{2} + \frac{1}{4}\sqrt{245 + 14\sqrt{70}}h), y_m - \frac{701}{684}hK_1 + \frac{5338}{2441}hK_2 - \frac{1675}{833}hK_3 + \frac{412}{685}hK_4).$$

Six Point Formula

The six points formula is of the form

$$y_{m+1} = y_m + h(\frac{141}{1646}K_1 + \frac{559}{3099}K_2 + \frac{2501}{10690}K_3 + \frac{2501}{10690}K_4 + \frac{559}{3099}K_5 + \frac{141}{1646}K_6) \quad (8)$$

where

$$K_1 = f(x_m + \frac{99}{2932}h, y_m + \frac{99}{2932}hf(x_m, y_m))$$

$$K_2 = f(x_m + \frac{269}{1588}h, y_m + \frac{269}{1588}hK_1)$$

$$K_3 = f(x_m + \frac{761}{1999}h, y_m - \frac{217}{853}hK_1 + \frac{1763}{2776}hK_2)$$

$$K_4 = f(x_m + \frac{1238}{1999}h, y_m + \frac{587}{573}hK_1 - \frac{248}{197}hK_2 + \frac{829}{971}hK_3)$$

$$K_5 = f(x_m + \frac{1319}{1588}h, y_m - \frac{2333}{1339}hK_1 + \frac{2153}{639}hK_2 - \frac{688}{431}hK_3 + \frac{1743}{2179}hK_4)$$

$$K_6 = f(x_m + \frac{372}{385}h, y_m + \frac{1033}{742}hK_1 - \frac{850}{487}hK_2 - \frac{734}{1017}hK_3 + \frac{1499}{3150}hK_4 + \frac{317}{2602}hK_5).$$

3 Example

There are three examples in this paper, the first two examples are the initial value problem of ordinary differential equations that are modified from the Newton Cooling Law by letting the derivative of the temperature of an object is in the form

$$T'(t) = k'(t)T(t) \quad (9)$$

and we are looking for the function $k(t)$ which $k(0) = 0$ and $\lim_{t\to\infty} k(t) = a$, $a = \log\left(\frac{S}{T_0}\right)$ where T_0 is the initial temperature of the object and S is the temperature of the surrounding. We select the function $k(t)$ of the form

$$k(t) = \frac{at}{b+t}. \quad (10)$$

With above assumption, we obtain the differential equation

$$T'(t) = \frac{abT(t)}{(b+t)^2} \quad (11)$$

with the initial condition

$$T(0) = T_0. \quad (12)$$

Example 1

The initial temperature is 82.3 the temperature of the surrounding is 24 and the temperature at time equals to 5 unit is 70.8. From these information, we obtain the equations

$$T'(t) = -\frac{44.28637763}{(t+35.93748011)^2}T(t) \quad (13)$$

with the initial condition

$$T(0) = 82.3. \quad (13)$$

The analytical solution of the equations (13)-(14) is $T(t) = 82.3e^{-\frac{1.232317277t}{t+35.93748011}}$. We use the formula (4), (5), (6), (7) and (8) to compute the numerical solution of the equation (11)-(12) and the results are in the following tables.

Formula	$h = 0.1$ $T(5) = 70.8$
(4)	70.80002266
(5)	70.79999993
(6)	70.79999995
(7)	70.79539569
(8)	70.80081804

Table 1

Formula	$h = 0.01$ $T(5) = 70.8$
(4)	70.80000017
(5)	70.79999995
(6)	70.79999995

(7)	70.79954004
(8)	70.80008270

Table 2

Formula	$h = 0.001$ $T(5) = 70.8$
(4)	70.79999995
(5)	70.79999995
(6)	70.79999995
(7)	70.79995484
(8)	70.80000921

Table 3.

Example 2

The initial temperature is 38 with the temperature of the surrounding is 81 and the temperature at the time 1 unit is 67. From these information, we obtain the following differential equation

$$T'(t) = \frac{0.253194807}{(t+0.3345465)^2} T(t) \quad (15)$$

with the initial condition

$$T(0) = 38 . \quad (16)$$

The analytical solution of the equations (15)-(16) is $T(t) = 38e^{-\frac{0.567106459t}{t+0.3345465}}$.

We use the formula (4), (5), (6), (7) and (8) to compute the numerical solution of the equation (15)-(16) and the results are in the following tables.

Formula	$h = 0.1$ $T(1) = 67$
(4)	67.08685152
(5)	67.00797686
(6)	66.99999922
(7)	66.94777057
(8)	67.08952211

Table 4

Formula	$h = 0.01$ $T(1) = 67$
(4)	67.00097963
(5)	67.00000882
(6)	66.99999922
(7)	66.94770566
(8)	67.00949389

Table 5

Formula	$h = 0.001$ $T(1) = 67$
(4)	67.00000976
(5)	66.99999988
(6)	66.99999987
(7)	66.99463388
(8)	67.00094834

Table 6.

In example 1, if we use the Newton Cooling Law then we obtain the following differential equation

$$T'(t) = 1.05474672 - 0.04394778T(t) \quad (17)$$

with the initial condition

$$T(0) = 82.3 . \quad (18)$$

We use the formula (4), (5), (6), (7) and (8) to compute the numerical solution of the equation (17)-(18) and the results are in the following tables.

Formula	$h = 0.1$ $T(5) = 70.8$
(4)	70.79908616
(5)	70.79906353
(6)	70.79906356
(7)	70.79259337
(8)	70.80020951

Table 7

Formula	$h = 0.01$ $T(5) = 70.8$
(4)	70.79906373
(5)	70.79906356
(6)	70.79906356
(7)	70.79841864
(8)	70.799179024

Table 8

Formula	$h = 0.001$ $T(5) = 70.8$
(4)	70.79906357
(5)	70.79990636
(6)	70.79906356
(7)	70.79899994
(8)	70.79907606

Table 9.

In example 2, if we use the Newton Cooling Law then we obtain the following differential equation

$$T'(t) = 1.05474672 - 0.04394778T(t) \quad (17)$$

with the initial condition

(18) $T(0) = 38$.

We use the formula (4), (5), (6), (7) and (8) to compute the numerical solution of the equation (17)-(18) and the results are in the following tables.

Formula	$h = 0.1$ $T(1) = 67.0$
(4)	63.96302242
(5)	63.98102975
(6)	63.98025846
(7)	64.22365745
(8)	63.98070181

Table 10

Formula	$h = 0.01$ $T(1) = 67.0$
(4)	63.98021840
(5)	63.98038339
(6)	63.98038268
(7)	64.00360208
(8)	63.98038172

Table 11

Formula	$h = 0.001$ $T(1) = 67.0$
(4)	63.98038112
(5)	63.98038276
(6)	63.98038276
(7)	63.98269310
(8)	63.98038139

Table 12.

Example 3

Find the numerical solution of the equation

(19) $$y'(x) = \frac{\sin x}{x^2} - \frac{2y}{x}$$

with the initial condition

(20) $y(2) = 1$.

The analytical solution of the equations (19)-(20) is

$$y(x) = \frac{4 + \cos(2) - \cos(x)}{x^2}.$$

We use the formula (4), (5), (6), (7) and (8) to compute the numerical solution of the equations (19)-(20) and the results are in the following tables.

Formula	$h = 0.1$ $T(1) = 0.5082057334$
(4)	0.50841101002
(5)	0.50820048372
(6)	0.50820603434
(7)	0.50434329143
(8)	0.50820248000

Table 13

Formula	$h = 0.01$ $T(1) = 0.5082057334$
(4)	0.50820705925
(5)	0.50820506847
(6)	0.50820507379
(7)	0.50782530248
(8)	0.50820510369

Table 14

Formula	$h = 0.001$ $T(1) = 0.5082057334$
(4)	0.50820509316
(5)	0.50820507329
(6)	0.50820507331
(7)	0.50816718524
(8)	0.50820510685

Table 15.

4 Conclusion

All above five new formulas work as good as they are expected. So the new five formulas will give us more freedom to select the way to look for the numerical solution of initial value problem of the ordinary differential equation. We strongly recommend the formula (4), (5) and (6). Note that in the example 2 which is the heating situation, the Newton Law of Cooling work quite different from our new approach. We will keep working on this kind of problem with other new function $k(t)$.

6 References

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