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Applying Differential Geometry to Kinematic Modeling in mobile Robotics

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Abstract: - We examine the kinematic model of a mobile robot with tools of differential geometry. These tools allow comprehensive modelling of even complex mobile articulated mechanical systems. Furthermore, they offer a very illustrative structure of the equations of motion by providing a so called trivial connection of pure motion and shape motion. Pure motion is the systems evolution in physical space, shape motion is the movement of the articulated mechanics such as wheels, fins, flaps, legs, etc. Different from the most other publications, we try to give a graphical interpretation of the complex mathematical objects as well as a detailed mathematical treatment of configuration spaces and its tangentials. This is motivated in the observation that people with an engineering background often find it difficult to get into this mathematical domain.

Key-Words: - manifold, geometric mechanics, mobile robot, Lie groups, fibre bundle, Ehresmann connection, principal connection, constraints

1 Introduction

By exploring mechanical systems from a differential geometric point of view, one wants to understand the structure of the equations of motion in a way, that helps to isolate the important objects which govern the motion of the system. The developments in the field of geometric mechanics have led to progress in the study of geometrical structure and dynamics of mechanical systems [4,17]. By modelling the locomotion process, it is possible to fully understand the behaviour of the system. An analysis of complex systems was made in [4]. A related modern example is the snakeboard made in [12] using simulations and experiments. Kelly and Murray [9] modelled many locomotive systems using kinematic constraints with results on controllability and motion generation, just as in [18] where the configuration variables are divided into two classes (shape and position) according to the basic structure of locomotion. In the 1970s Brockett put the theory of Lie groups and their Lie algebras in the context of nonlinear control. For motion control problems involving rotating and translating bodies such as mobile robots, space systems, underwater vechicles, the natural appearance of certain Lie groups derives from the fact that they describe the configuration space of the system or a part of it [10]. For an introduction to robotics in a mathematical sense [15] is recommended and for the understanding of mathematical notions we suggest [11,1,19,21,3]. Furthermore in this paper we try to give a graphical and concrete description of the mathematical notions used in modelling and reducing the equations of motion of a mobile robot. We try to clarify aspects that are hidden by the mathematical framework, i.e. the power of differential geometric formalism, from a different point of view. This formalism may not seem very familiar to engineers and robotic practitioners, thus we try to make it understandable, giving aid in the continued study of robotic locomotion.

The organisation of this paper is as follows: after the short overview in chapter 1, in chapter 2 we handle some basics for working with mechanical systems in a graphical way, followed by an introduction of some matrix groups that have great significance in robotics, as well as other useful tools and concepts. We will apply this theory to a simple two wheeled robot. In chapter 3, we continue to study the robot in motion also in a graphical way. We explain some basics about Lie groups and their Lie algebras.

2 Basics of geometric mechanics

2.1 The configuration space is a manifold

For mechanical systems composed of rigid bodies, the *configuration* is the collection of all 'position' variables. The number of variables n is equal to the number of degrees of freedom (DoF) of the mechanical system. The set of all configurations is called *configuration space* Q. The vector $\mathbf{q}=(q_1,\dots,q_n)\in Q$ denotes a configuration with generalized positions q_i. Fig.1 illustrates some simple mechanics and their configuration spaces. For a rigid planar rod pendulum (top left) the configuration space is $S^1=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$. The spot on the S^1 circle represents the current configuration. However, a configuration space at first is no mathematical object one can calculate on. Thus one needs an object from mathematics which is comparable to configuration spaces. Consider a topological manifold which is a topological space and locally looks like the euclidian $\mathbb{R}^n$ [1]. Properties of $\mathbb{R}^n$ can be transfered to manifolds, in particular one can use calculus on manifolds. Note that a manifold has no global, but only local coordinates. Such a manifold is isomorphic to configuration spaces of many mechanical systems. Generally any q_i is of unlimited range[2]. Thus one can speak of the *configuration manifold*.

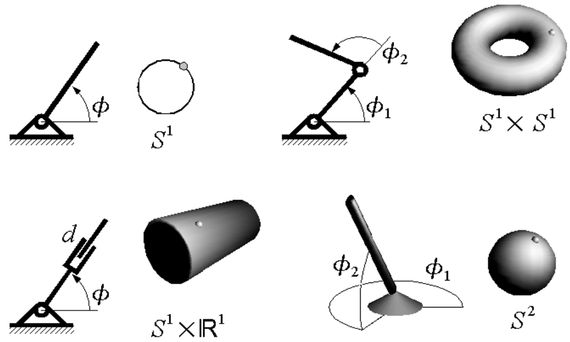

Fig.1 – Simple mechanics and their configuration manifolds.

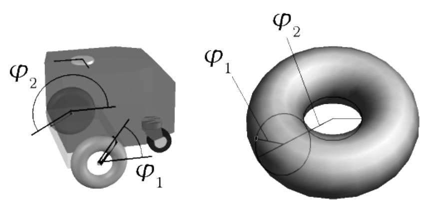

Fig.2 – Part of configuration manifold of mobile robot.

[1]more precisely a manifold is a space where for any point there exists an open neighbourhood U and a homeomorphism $\phi: U\to\phi(U)\subset\mathbb{R}^n$ on any open set $\phi(U)\subset\mathbb{R}^n$. Refer to e.g. [11,16] for exact definitions.

[2] the range of q_i is often limited in practice but this would make theory more complex. It seems easier to omit limits in theory and consider them in practice – if needed.

The articulated mechanics of the considered mobile robot consist of two independently driven wheels which form a $S^1\times S^1$ part of the configuration manifold as shown in Fig.2. This is also called *shape space* and φ_1,φ_2 are called *shape variables*. Since the robot is free to locomote there is a second part which represents the robots position[3] on a plane and will be introduced in the next subsection.

2.2 Special matrix groups are manifolds

If a body on a horizontal plane is free to rotate about the vertical axis, the orientation $\theta\in\mathbb{R}$ can be represented as a matrix

$$\mathbf{R}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

$\mathbf{R}$ is orthogonal i.e. $\det(\mathbf{R})=1$ (in a right-handed coordinate frame) and $\mathbf{R}^T=\mathbf{R}^{-1}$ [4]. Hence $\mathbf{R}$ is a member of the special orthogonal group $SO(2)$.

The mobile robot is movable on a plane and its position can also be written in matrix form:

$$g=\begin{bmatrix}\cos\theta & -\sin\theta & x\\ \sin\theta & \cos\theta & y\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\mathbf{R} & p\\ \mathbf{0} & 1\end{bmatrix}. \tag{1}$$

Fig.3 shows g of two example positions: g_0 at the origin of the plane coordinate frame and g_1 at another position. All g form the special euclidian matrix group $SE(2)$. These matrices have the properties $\mathbf{R}\in SO(2)$ and $p=(x,y)\in\mathbb{R}^2$. Fig.3 also gives an illustration of $SE(2)$. Locally it looks like euclidian $\mathbb{R}^3$, but globally it is a structure in $\mathbb{R}^4$. One may describe it as an infinitesimally long tube with an infinitesimally thick wall.

A benefit of using $SE(2)$ is, that its members not only describe positions, but also translations. The matrix in (1) describes a translation about (x,y) and a rotation about θ [5]. When looking at the properties of

[3] There is sometimes confusion about the terms “orientation”, “position”, “location”, etc. Throughout this paper, we use them as follows: **location** = where a point is, **orientation** = where an object is directed to, **position** = location and orientation of an object in space.

[4] the columns of a rotation matrix are orthonormal. This yields to $\mathbf{R}\mathbf{R}^T=\mathbf{R}^T\mathbf{R}=\mathbf{I}\ \Rightarrow\mathbf{R}^T=\mathbf{R}^{-1}$.

[5] We use the following terms for rigid body motion: **rotation** = act of turning an object about its axes, **translation (shift)** = a uniform movement without rotation (WordNet), **motion** = rotation and translation. Although at first sight, these terms

a differentiable manifold, one finds that they also hold for $SE(2)$, because the general matrix group $GL(n)=\{\mathbf{R}\in\mathcal{M}_{n\times n}(\mathbb{R}): det\,\mathbf{R}\neq 1\}$ is the supergroup of $SE(2)$ and is also a differentiable manifold. And $GL(n)$, the group of linear isomorphisms from $\mathbb{R}^n$ onto $\mathbb{R}^n$, is a differentiable manifold, because it is open in vector space $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ [6] (it can be seen as the inverse image of $\mathbb{R}\backslash\{0\}$ in relation to the determinant map, i.e. $GL(n)=det^{-1}(\mathbb{R}\backslash\{0\})$). With a fixed base in $\mathbb{R}^n$ we have $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)\approx\mathcal{M}_{n\times n}(\mathbb{R})$. The properties of $SE(2)$ yield to the continuous and differentiable transformation

$$\mathbb{R}^3\ni(x,y,\theta)\rightarrow\begin{bmatrix}\cos\theta & -\sin\theta & x\\ \sin\theta & \cos\theta & y\\ 0 & 0 & 1\end{bmatrix}\in SE(2)\,.$$

Thus one can combine $SE(2)$ with the manifold part of the articulated mechanics in order to obtain the entire configuration manifold of the mobile robot which is done in the next subsection.

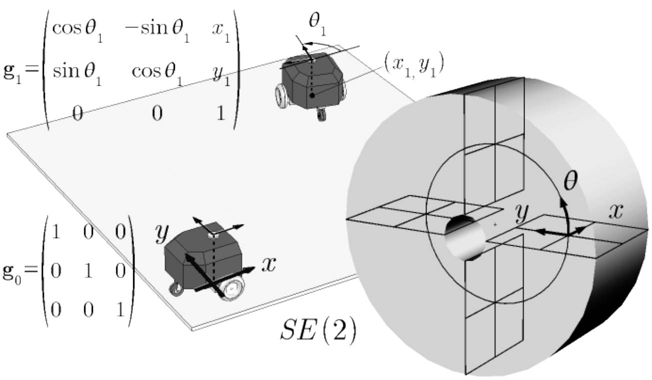

Fig.3 – Matrix group $SE(2)$ for describing positions

2.3 Bundled manifolds

When combining the two parts of the manifold we introduced in the preceding subsections, one obtains the configuration manifold of the mobile robot as

$$Q=SE(2)\times S^1\times S^1\,. \tag{2}$$

The two S^1 belong to the angles of the two wheels. Equation (2) shows a natural decomposition of Q in the sense that $SE(2)$ is the position where we want to move to, and $S^1\times S^1$ is with what 'tools' we have to achieve that. $SE(2)=G$ is called *fibre* and $S^1\times S^1=M=Q/G$ is called *base*. At every base element an orthogonal fibre is attached. All fibres together form a fibre bundle[7], which in turn is Q. Fig.4 illustrates the configuration manifold of the mobile robot as a fibre bundle. Note that every fibre string is a 4-dimensional manifold object (see Fig.3). Another illustration is given in Fig.5: the fibres are bundled in the sense that the height of every 'bundle-ring' above the base represents a unique position of the robot in the plane. Just as in Fig.3 two example positions are shown in the figure: the zero position g_0 and another position g_1. Note that a bundle-ring again is the entire torus surface since at every g the two wheels can have arbitrary angles φ_1,φ_2.

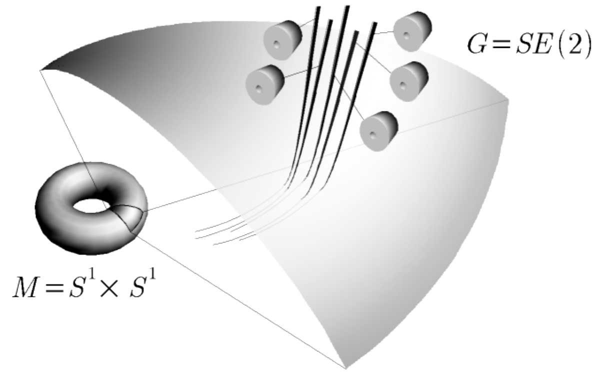

Fig.4 – Configuration manifold of the robot as a fibre bundle

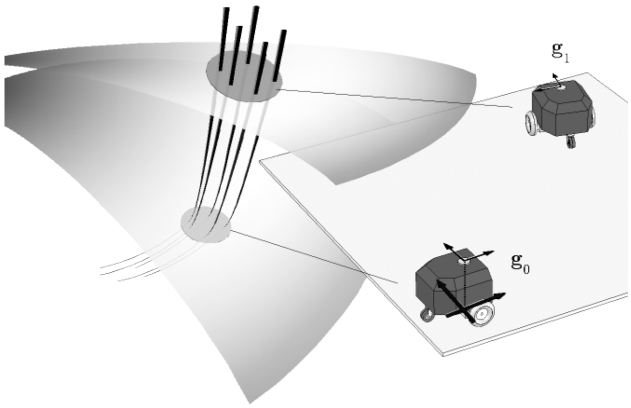

Fig.5 – Representations of different positions on the bundled configuration manifold

may give the impression of observing a continuous motion, we mainly use them without considering the evolution of motion in time.

[6]the set of all linear maps from $\mathbb{R}^n$ into $\mathbb{R}^n$.

[7]A fibre bundle is a simple geometric structure consisting of a base space and a fibre space. More precisely it is a triplet of the manifolds E (total space), M (base space) and the projection $\pi: E\rightarrow M$. For detailed definitions refer to e.g. [6].

2.4 Mechanical constraints

Most mechanical systems have constraints which reduce the number of DoF. One way to characterise constraints is to distinguish between holonomic and nonholonomic constraints. For a better understanding of different constraints refer to [13]. Any rolling motion without sliding is nonholonomic (non integrable) which is also the case for the mobile robot. In detail the constraints are

$$\begin{aligned}\dot{x}&=(\rho/2)\cdot\cos\theta\cdot(\dot{\varphi}_1+\dot{\varphi}_2)\\ \dot{y}&=(\rho/2)\cdot\sin\theta\cdot(\dot{\varphi}_1+\dot{\varphi}_2)\\ \dot{\theta}&=(\rho/(2w))\cdot(\dot{\varphi}_1-\dot{\varphi}_2)\end{aligned} \tag{3}$$

where ρ is the radius of a wheel and w is the axial distance between the wheels. (3) shows no constraint on the shape variables i.e. from a starting $(\varphi_1(t_0),\varphi_2(t_0))$, the system is free to move anywhere on the torus surface (Fig.6 left). Whereas an interesting question is what impact the constraints have on the position in $SE(2)$. This is answered by (3) and some physical deliberations. At $\theta=0$ and $\theta=\pi$ the robot may only move along x-, and at $\theta=\pi/2$ and $\theta=3\pi/2$ it may only move in y-direction. Cutting this out of the $SE(2)$ manifold as shown in Fig.3, one obtains a structure we call the *twisted band* manifold. Fig.6 (right) gives an illustration. This manifold structure is true for all ground-moving mechanics which have constraints that do not allow sideway movements.

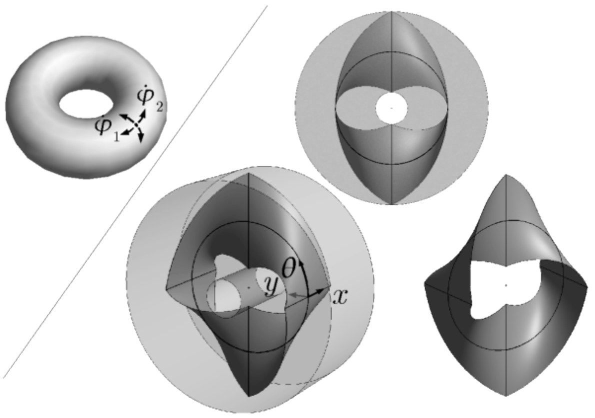

Fig.6 – Images of the constrained configuration manifold

3 Tangent space and tangent bundle

Now we are going to examine the mobile robot in motion, i.e. there is a $\dot{\mathbf{q}}\neq 0$. There are different ways how to introduce tangent vectors. They can either be defined by coordinate representation or as velocity vectors of a curve on manifolds. The set of all velocity vectors at $q\in Q$ belongs to a vector space assigned to q. This is called the *tangent space* $T_q Q$. Naturally it has the same dimension as Q. For an interval $I\subset\mathbb{R}$ and a curve $c:I\to Q$ the tangent vector (velocity vector) of c at time $t_0\in I$ is

$$\dot{c}(t_0):=T_{t_0}c\frac{d}{dt}\Big|_{t_0}\in T_{c(t_0)}Q\,.$$

With aid of the tangent space one can transfer the derivation from real calculus to manifolds. From the unity of tangent spaces at all points on a manifold, one can create the *tangent bundle* $TQ=\cup_{q\in Q}T_q Q$. The tangent bundle TQ of a manifold Q also is a fibre bundle with fibres $F=T_q Q$, base Q, and the projection $\pi:TQ\to Q$. Every base point is traversed by a fibre. Fig.7 shows how TQ looks like for one of the wheels of the robot. When it rotates with constant speed ω_1, we are moving on TQ at constant 'height' above Q.

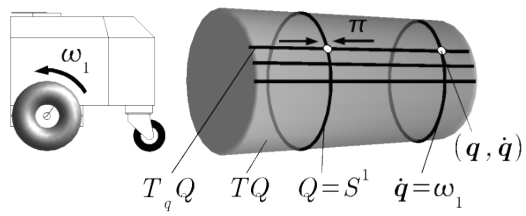

Fig.7 – Tangent space, tangent bundle and natural projection of one wheel

A vector field on a manifold Q is a part (cross-section) of the tangent bundle TQ. In other words: a vector field assigns a tangent vector to every point $q\in Q$. One can interpret a vector field as the right side of a system of ordinary first order differential equations.

3.1 Lie groups and Lie algebras

Lie groups are shown to be useful in robot kinematics and control, thus we are going to introduce the basic concepts. A *Lie group*[8] is a manifold, which additionally has a group structure which is compatible with the structure of the manifold. Important examples

[8]a space G is called Lie group, if: a) $(G,\cdot)$ is a group; b) G is a n-dimensional differentiable manifold; c) the mappings $G\times G\to G$, $(g,h)\to gh$ and $G\to G$, $g\to g^{-1}$ are smooth.

are the matrix groups, which are also Lie groups. One may refer to [6] for detailed proofs for the general linear group $GL(n)$ (all real-valued nonsingular $n\times n$ matrices), and the special orthogonal and euclidian groups $SO(2), SO(3), SE(2)$. For the mobile robot we will look closer at $SE(2)$ in order to better understand the structure of a Lie group. One may write $SE(2)$ as $\mathbb{R}^2\times SO(2)$. Since $SO(2)$ and $\mathbb{R}^2$ are Lie groups, $SE(2)$ is as well a Lie group, and $\dim SE(2)=\dim SO(2)+\dim\mathbb{R}^2=3$. An element $g=(p,\mathbf{R})\in SE(2)$ has the coordinates $g=(x,y,\theta)$ with $(x,y)\in\mathbb{R}^2$ and $\theta\in SO(2)$. g, written as a 3×3-matrix, is

$$g=\begin{bmatrix}\cos\theta & -\sin\theta & x\\ \sin\theta & \cos\theta & y\\ 0 & 0 & 1\end{bmatrix}.$$

$SE(2)$ may be identified with the space of all 3×3-matrices of the form

$$g=\begin{bmatrix}\mathbf{R} & p\\ 0 & 1\end{bmatrix},\ \mathbf{R}\in SO(2),\ p\in\mathbb{R}^2$$

while

$$g=(p,\mathbf{R})\rightarrow g=\begin{bmatrix}\mathbf{R} & p\\ 0 & 1\end{bmatrix}.$$

The group structure is given by

$$g\cdot h=(p_1,\mathbf{R_1})(p_2,\mathbf{R_2})=(\mathbf{R_1}p_2+p_1,\mathbf{R_1}\mathbf{R_2})\ \text{and}$$
$$g^{-1}=(p,\mathbf{R})^{-1}=(-\mathbf{R^T}p,\mathbf{R^T})\cdot$$

Written in coordinates ($SE(2)$ as a vector space):

$$g\cdot h=(\cos\theta_1 x_2-\sin\theta_1 y_2+x_1,$$
$$\cos\theta_1 x_2+\sin\theta_1 y_2+y_2,\theta_1+\theta_2)\ \text{and}$$
$$g^{-1}=\left(-\cos\theta\, x-\sin\theta\, y,\sin\theta\, x-\cos\theta\, y,-\theta\right).$$

As one can see, the two operators are differentiable (differentiable in their components).

To a Lie group G one can assign a *Lie algebra* $\mathcal{G}$. $\mathcal{G}$ is nothing else but a collection of vector fields isomorphic to the tangent space at the identity T_eG. Studying the Lie algebra one can derive properties of the structure of the associated Lie group. With these properties one can define the *exponential map* of a Lie group, which in turn is the link between Lie algebra $\mathcal{G}$ and Lie group G. Since a Lie group in general is not abelian ($g\cdot h\neq h\cdot g$), one may introduce a left translation $L_g: G\rightarrow G$ as $L_g(h)=g\cdot h$ and a right translation $R_g: G\rightarrow G$ as $R_g(h)=h\cdot g$, which commute ($L_g\circ R_g=R_g\circ L_g$). The left or right translation of a Lie group can be seen as an action[9], which is of certain interest for the robot kinematics. We can imagine actions as a motion of the robot in the plane from one position to another.

In the previous section we have introduced the tangent bundle and the tangent space of some simple mechanical systems. Now we are going to return to our robot where we want to examine the Lie group part $G\subset Q$. Whenever we have actual or desired motions of any non-stationary rigid mechanics, we are working on a Lie group and its tangent bundle. We have seen that Lie group elements describe both, positions and translations. With the Lie group structure we can express tangent vectors in T_gG with elements of the associated Lie algebra. The corresponding vector fields are the set of all left invariant vector fields (i.e. $g\in G: T_hL_gX(h)=X(g\cdot h)$, for all $h\in G$, so it does not vary under any left action). Like vector spaces the set of all left-invariant vector fields (which are also differentiable, for more details see [21,2]) and T_eG are isomorphic, where e is the identity element. Thus if we know X at the identity e, we know it on the entire G ($X(g)=T_eL_gX(e)$). If we write elements of T_eG with ξ,η, then the Lie algebra structure is defined with help of the Lie bracket (for the definition refer to e.g. [19,13]): $[\xi,\eta]=[X,Y](e),\forall\xi,\eta\in G$. Fig.8 gives an illustration of G, its identity e, a left invariant vector field X, and elements we have discussed so far.

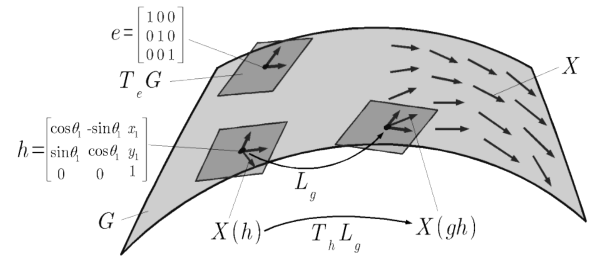

Fig.8 – $SE(2)$ Lie group with a left-invariant vector field X

[9]an action is a differentiable map $\Phi: G\times M\rightarrow M$ so that

$$\Phi(e,p)=p,\forall p\in M,\ \text{and}$$
$$\Phi(g,\Phi(h,p))=\Phi(g\cdot h,p),\forall p\in M,\forall g,h\in G$$

where G is a Lie group and M is a manifold.

3.2 The exponential map

The exponential map $\exp: G \to G$ links the Lie algebra with its Lie group. It is a local diffeomorphism (i.e. a smooth map with its inverse also smooth). In the following we will study Lie algebra and exponential map of the Lie group $SE(2)$. Further detailed proofs, properties, and applications of exponential map, actions, and other useful tools of geometric mechanics can be found in [6, 21,13]. The Lie algebra of $SE(2)$, denoted as $se(2)$, is $se(2) = so(2) \times \mathbb{R}^2$, and can be identified with 3×3 matrices

$$\hat{\xi} = \begin{bmatrix} 0 & -\xi_3 & \xi_1 \\ \xi_3 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix}^{10}.$$

Every element of $SE(2)$ one can write as an element of the Lie algebra with the exponential map:

$$\exp(\xi) = \Big(\frac{\xi_1}{\xi_3}\sin\xi_3 + \frac{\xi_2}{\xi_3}(\cos\xi_3 - 1), \\ \frac{\xi_2}{\xi_3}\sin\xi_3 + \frac{\xi_1}{\xi_3}(1-\cos\xi_3), \xi_3\Big).$$

With the left translation one can describe the velocity on a trajectory $g(t) \in G$ with a single Lie algebra element. The body- and spatial velocity of a rigid body are $\xi^b = g^{-1} \cdot \dot{g}$ and $\xi^s = \dot{g} \cdot g^{-1}$ respectively. The physical interpretations are the translatory and rotatory velocity of an object with respect to a body- and a spatial coordinate frame. The relation between both velocities is described with the *lifted adjoint action* (details e.g. in [2]). For $SE(2)$ the lifted adjoint action is

$$Ad_g = T_e\left(R_{g^{-1}} L_g\right) = \\ = T_g R_{g^{-1}} T_e L_g = \begin{bmatrix} \cos\theta & -\sin\theta & y \\ \sin\theta & \cos\theta & x \\ 0 & 0 & 1 \end{bmatrix}.$$

[10] as a vector space $se(2)$ is isomorphic with $\mathbb{R}^3$ by

$$\hat{\xi} = \begin{bmatrix} 0 & -\varpi & v_1 \\ \varpi & 0 & v_2 \\ 0 & 0 & 0 \end{bmatrix} \to \xi = (\varpi, v) \in \mathbb{R}^3.$$

This isomorphism also is a Lie algebra isomorphism by $[\xi, \eta] = \hat{\xi}\hat{\eta} - \hat{\eta}\hat{\xi}$.

3.3 Connections

As outlined above one can describe positions of the robot with special Lie groups, which in turn are part of the configuration manifold. For control problems we are interested in the other part of the configuration, the shape variables. The shape space is Q/G, also called reduced space. Lie group and shape space create the fibre bundle. When speaking of a *trivial bundle* one can span the configuration manifold with the base Q/G and the fibres G attached to every $r \in Q/G$. In this context one may introduce the so called *Ehresmann connection*. An Ehresmann connection A is a vector-valued one-form on Q, so that $A_q: T_q Q \to Ker\, T_q \pi$ is linear $\forall q \in Q$ and it is a projection. The kernel of A_q is called horizontal space and $Ker\, T_q \pi = V_q$ is called vertical space. This means that one can split the tangent space $T_q Q$ into a horizontal and a vertical part. The tangent vector is projected onto its vertical part by a connection[11] or principal connection[12] (for details on connections refer to e.g. [2]). One finds that the vertical space is tangent to the vertical fibre above $q \in Q$, i.e. all points which are mapped onto the same point (for details refer to [6]).

3.4 Constrained distribution

The *constrained distribution* consists of all tangent vectors which meet the constraints. A distribution is a collection of tangent spaces at a point on a manifold, so it is a partial bundle of its tangent bundle. In other words a distribution can be represented with linearily independent differentiable vector fields. If one writes the constraints as oneforms $\omega_1, \omega_2, \omega_3$, the distribution is $\mathcal{D}_q = \mathrm{Ker}\{\omega_1, \omega_2, \omega_3\}$. The tangent space $T_q Q$ can be split uniquely into a vertical part $V_q = T_q Orb_q = \{\xi_Q^q, \xi \in se(2)\}$ [13], and a horizontal part, which in case of the mobile robot is equal to the constraints (3). Then we have $V_q \cap \mathcal{D}_q = \{0\}$ (which is not valid for other mechanical systems). In other words, the vertical space represents pure motions of the robot on the plane without shape motions, and the horizontal space represents shape motions, i.e. turning

[11] a connection is an allocation of horizontal spaces $H_q \subset T_q Q$ for every $q \in Q$, so that $T_q Q = H_q Q \oplus V_q Q$, $T_q \Phi_g H_q Q = H_{gq} Q$ for all $q \in Q$ and $g \in G$ and $H_q Q$ depends differentiably on q.

[12] a principal connection is a connection like a Lie algebra valued oneform on Q.

[13] $\xi_Q^q = d/dt\, \Phi_{\exp(t\xi)}(q)\big|_{t=0}$ is the infinitesimal generator according to an action $\xi \in G$.

wheels, without pure motions. The connection describes the link between them.

4 Conclusion and Outlook

In this paper we explained the basics of the mathematical framework necessary to model a nonholonomic mechanical system (that includes almost all mobile robots presently in use) in a way that divides the configuration variables in shape and position classes. Here we only treated the pure kinematic case [4], where the group variables do not interact with the shape variables (the dynamics of the system are constrained using only configuration velocities). First we choose the position variable (describing the position and orientation of the robot with respect to an inertial frame). The remaining variables build the shape of the system and its variation induces the locomotion. The relationship between shape and position changes is made by a mathematical construction named connection.

There are some systems where the interaction between constraints and the group action is not trivial, producing momentum changes which result in locomotion [18,4]. This interaction plays an important role in defining a connection. The mathematical properties of the connection allow to simplify results for both dynamics and control of locomotion systems. [7] and [20] have lead to an understanding of the reduction process [4,18] building the mechanical connection.

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