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Stability of quasi-periodic orbit in Discrete Recurrent Neural Network

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Abstract: - A simple discrete recurrent neural network model is considered. The local stability is analyzed with the associated characteristic model. In order to study the quasi-periodic orbit dynamic behavior, it is necessary to determinate the Neimark-Sacker bifurcation. In the case of two neurons, one necessary condition that produces the Neimark-Sacker bifurcation is found. In addition to this, the stability and direction of the Neimark-Sacker are determined by applying the normal form theory and the center manifold theorem. An example is given and numerical simulation are performed to illustrate the obtained results. The phase-locking is analyzed given some experimental result of Arnold Tongue in determinate weight configuration.

Key-Words: - Nonlinear System, Neural Networks, Bifurcation, Neimark Sacker, Arnold Tongue, Quasi-periodic orbit.

1 Introduction

The purpose of this work is to present some results related to the analysis of the dynamics of a discrete recurrent neural network. The particular model of network in which we are interested is the Williams-Zipser network, also known as Discrete-Time Recurrent Neural Network (DTRNN) in [8]. Its state evolution equation is

$$x_i(k+1) = f\left(\sum_{n=1}^{N} w_{in} x_n(k) + \sum_{m=1}^{M} w'_{im} u_m(k) + w''_i \right) \quad (1)$$

where

$x_i(k)$ is the ith neuron output

$u_m(k)$ is the mth input of the network

w_{in}, w'_{im} are the weight factors of the neuron outputs, network inputs and w''_i is a bias weight.

$f(\cdot)$ is a continuous, bounded, monotonically increasing function such as the hyperbolic tangent.

From the point of view of the dynamical theory it is interesting to study the equilibrium or fixed points. These points do not change in time. Their character or stability is given by the local behavior of nearby trajectories. A fixed point can attract (sink), repel (source) or have directions of attraction and repulsion (saddle) of close trajectories [5]. Next in complexity are periodic trajectories, quasi-periodic trajectories or even chaotic sets, each with its own stability characterization. All these features are similar in a class of topologically equivalent systems [2]. When a system parameter is varied the system can reach a critical point in which it is no longer equivalent. This is called a bifurcation [12], and the system will exhibit new behaviors.

With respect to discrete recurrent neural networks as systems, several results about their dynamics are available in the literature. The most general result is derived using the Lyapunov stability theorem in Marcus and Westervelt [1] and it establishes that for a symmetric weight matrix there are only stable equilibrium states. These are fixed points and period two limit cycles. In this paper, are also given conditions under which there are only fixed-point attractors are given. More recently, Cao [7] has proposed another condition less restrictive and more complex. Wang [11] describes one interesting type of trajectories, the quasi-periodic orbits. Passeman [4] obtains some experimental results such as the coexisting of the periodic, quasi-periodic and chaotic attractors. In other hand, In [9] give the position, number and stability types of fixed points of a two-

neuron discrete recurrent network with nonzero weights are investigated.
There are some works that analyse the hopfield continous neural networks [6, 3] like [7, 10], in this paper shown the stability of hopf-bifurcation with two delays.
This paper is divided into three sections. In the section 2, the local stability of the Williams-Zipser neural network and the necessary condition that generate the Neimark-Sacker bifurcation are analysed. In the section 3, the condition of the stability of the bifurcation is established. We can conclude that it depends on the derivatives relation of activation function on zero coordinate. In the section 4, some simulations of quasi-periodic orbit with the tangent hyperbolic as activation function are shown. In the last function section we explain the Arnold tongue, and we calculate some simulation with four different configuration, and we show that the periodic orbit has the similar probably like the quasi-periodic orbit.

2 Local Stability Analysis

In the development below two-neurons neural network are considerate. It is usual that the activation function is a sigmoid function or tangent hyperbolic function. In order to simplify the notation we redefine (x_1, x_2) as (x, y).
Firstly, the analytical condition of fixed point can be shown.

$$x = f(w_{11}x + w_{12}y) \qquad (2.a)$$

$$y = f(w_{21}x + w_{22}y) \qquad (2.b)$$

In this mapping the elements of the jacobian matrix in the fixed point (x, y) are

$$A = \begin{bmatrix} w_{11}f'(f^{-1}(x)) & w_{12}f'(f^{-1}(x)) \\ w_{21}f'(f^{-1}(y)) & w_{22}f'(f^{-1}(y)) \end{bmatrix} \qquad (3)$$

The associated characteristic equation of the linearized system evaluated in the fixed point is

$$\lambda^2 - \left[w_{11}f'(f^{-1}(x)) + w_{22}f'(f^{-1}(y))\right]\lambda + |W|f'(f^{-1}(x))f'(f^{-1}(y)) = 0 \qquad (4)$$

where w_{11},w_{22} and $|W|$ are the diagonal elements and the determinant of the matrix weight, respectively.
We can define new variables

$$\sigma_1 = \frac{w_{11}f'(f^{-1}(x) + w_{22}f'(f^{-1}(y))}{2} \qquad (5)$$

$$\sigma_2 = |W|f'(f^{-1}(x))f'(f^{-1}(y)) \qquad (6)$$

The eigenvalues of the characteristic (4) are defined as

$$\lambda_\pm = \sigma_1 \pm \sqrt{\sigma_1^2 - \sigma_2} \qquad (7)$$

The Neimark-Sacker bifurcation appears when two complex conjugate eigenvalues reach the unit circle. It is easy to show that the limit conditions are

$$\lambda_\pm = e^{\pm i\theta_0} \qquad (8)$$

where

$$\theta_0 = \text{arctang}\left(\frac{\sqrt{1-\sigma_1^2}}{\sigma_1}\right)$$

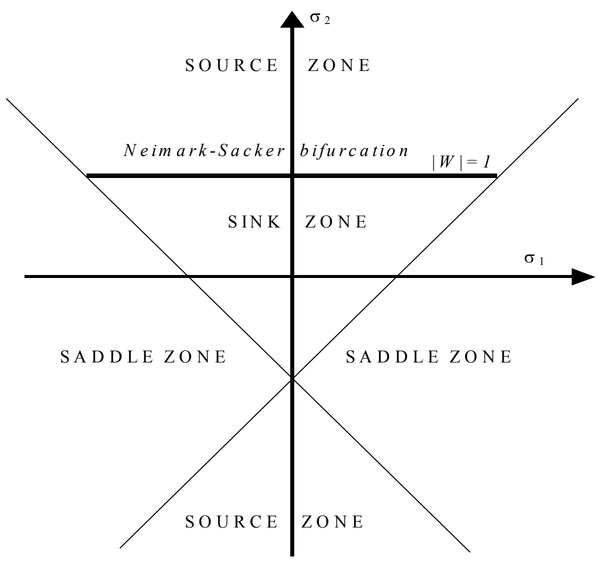


Fig. 1. The stability regions and the bifurcation lines in the fixed point (0, 0).

The boundaries between the regions shown in Fig. 1 are the bifurcations, that is to say, the limit zones where the fixed point changes its character. The *Neimark-Sacker* bifurcation represented by the line $|W|$=1 in Fig. 1. The necessary condition that produce the Neimark-Sacker bifurcation can be stated.

Lemma 1. Suppose that activation function $f \in C^2$, bounded, monotonically increasing . Then Neimark-Sacker bifurcation only appears when $w_{12}w_{21}<0$.

Proof:
The methodology to prove the theorem is considerate the constricted equation corresponding to the Neimark-Sacker bifurcation and extract it a condition respect to the non diagonal weights. Following it is analysed the different share of the F function considering the non diagonal weights condition and we will calculate the different configuration respect to the number of fixed points.
Assume that one fixed point (x_0, y_0) produce the Neimark-Sacker bifurcation then
$|W| X_0 Y_0 = 1$

$$-2 < \| w_{11} X_0 + w_{22} Y_0 \| < 2 \qquad (9)$$

where || || is the absolute value and
$X_0 = f'(f^{-1}(x_0)) \quad Y_0 = f'(f^{-1}(y_0))$

Taking into account the last expressions, the limit curves in the plane (X_0, Y_0) are

$$Y_c = \frac{1}{|W| X_0} \qquad (10)$$

$$Y_{\pm 2l} = \frac{\pm 2 - w_{11} X_0}{w_{22}} \qquad (11)$$

If it is consider the bifurcation condition in (10) and the fact that f is monotonically increasing |W| is positive, because of this, the curves present the behavior in Fig. 2. In order to obtain the curves intersection points, the following second order equation can be stated

$$X_0^2 - \frac{2}{w_{11}} X_0 + \frac{w_{22}}{w_{11} |W|} = 0 \qquad (12)$$

This it only possible if $w_{12}w_{21}<0$. □

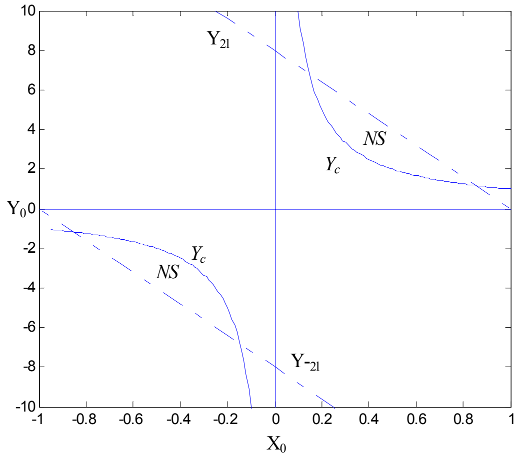


Fig. 2. Curves that define the bifurcation Neimark-Sacker conditions with $|W|=1$, $w_{11}=2$, $w_{22}=0.25$.

3 Direction and stability of bifurcating quasi-periodic orbit

In order to determinate the direction and stability of Neimark-Sacker bifurcation it is necessary to use the center manifold theory [12]. The center manifold theory demonstrate that the mapping behavior in the bifurcation is the complex mapping below

$$\tilde{z} = e^{i\theta} z(1 + d(0)|z|^2) + O(|z|^4) \qquad (13)$$

The parameter a(0) is [12]

$$\begin{aligned} a(0) = \operatorname{Re}(d(0)) = \frac{1}{2} \operatorname{Re}\{ e^{-i\theta} [\langle p, C(q,q,\bar{q}) \rangle \\ + 2\langle p, B(q,(E-A)^{-1} B(q,\bar{q})) \rangle \\ + 2\langle p, B(\bar{q},(e^{2i\theta} E - A)^{-1} B(q,q)) \rangle \} \end{aligned} \qquad (14)$$

where E is the identity matrix, B and C are the second and third derivative terms of the mapping Taylor development, respectively, and p, q are the eigenvector Jacobian matrix and its transpose, respectively. These vectors satisfy the normalization condition

$$\langle p, q \rangle = 1 \qquad (15)$$

The zero in the argument of the coefficient a represents the critical parameter of the system where produce the bifurcation takes place. The *a*(0) sign determinate the bifurcation direction. When *a*(0) is positive an unstable quasi-periodic orbit (transition subcritical) disappears, and in opposite case, appears a stable quasi-periodic orbit (transition supercritical).

In the neural network mapping, p and q are

$$q = \left\{1, \frac{e^{i\theta} - w_{11} f'(0)}{w_{12} f'(0)}\right\} \quad (16)$$

$$p = \beta\left\{1, \frac{e^{-i\theta} - w_{11} f'(0)}{w_{21} f'(0)}\right\} \quad (17)$$

where

$$\beta = \frac{[2 Sin\ \theta_0 + i(w_{11} - w_{22}) f'(0)]}{4 Sin\ \theta_0}$$

The Taylor development terms are

$$B_i(q,q) = \sum_{j,k=1}^{2} \frac{\partial f_i}{\partial x_j \partial x_k} q_j q_k = f''(0) \sum_{j,k=1}^{2} w_{ij} w_{ik} q_j q_k \quad (18)$$

$$C_i(q,q,\overline{q}) = \sum_{j,k,l=1}^{2} \frac{\partial f_i}{\partial x_j \partial x_k \partial x_l} q_j q_k \overline{q}_l = f'''(0) \sum_{j,k,l=1}^{2} w_{ij} w_{ik} w_{il} q_j q_k \overline{q}_l \quad (19)$$

We assume that the second derivative of the activation is zero as the hyperbolic tangent function. Consequently, $B \equiv 0$ and it is only necessary to calculate the third derivative evaluated in the eigenvector q and it conjugate. Using the expressions (14), (16), (17), and the critical eigenvalues, the relation below can be obtained

$$a(0) = \frac{f'''(0)}{4 w_{12}^2 f'(0)^5}\left(1 - f'(0)^2 w_{11} w_{22} + w_{12}^2 f'(0)^2\right) \quad (20)$$

with the equation (9)

$$a(0) = \frac{f'''(0)}{4 f'(0)^3}\left(1 - \frac{f'(0)^2 w_{21}}{w_{12}}\right) \quad (21)$$

Taking into account the necessary bifurcation condition $w_{12}w_{21}<0$, the quasi-periodic orbit stability only depends on the relation between the sign of the first and third derivatives of activation function evaluated in zero. When the activation function is the hyperbolic tangent, the bifurcation is supercritical and the quasi-periodic orbit is always stable.

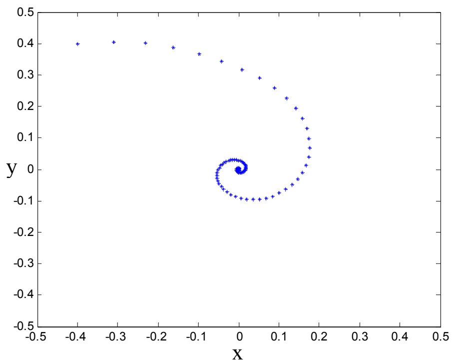
4 Simulation

In general, the follow matrix weight is used

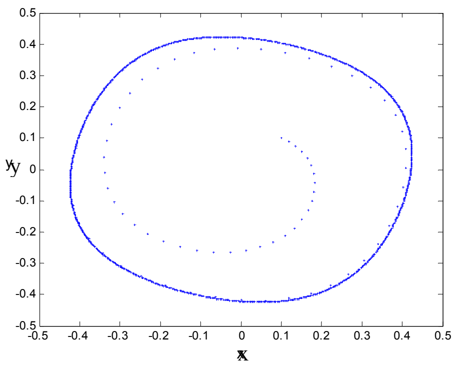
$$W = (1-\alpha)\begin{bmatrix} \cos\theta_0 & sen\theta_0 \\ -sen\theta_0 & \cos\theta_0 \end{bmatrix} \quad (22)$$

With the matrix above the parameter α produces the Neimark-Sacker bifurcation when it changes from a negative value to a positive one.

In order to show the application of the result that has been obtained, the hyperbolic tangent function is considered. This function is monotonically increasing, the coordinate in the origin and second derivative are zero. Using the result of the previous section it can be said that the quasi-periodic orbit is always stable independently on the parameters. As an example, consider the weight matrix (22) with $\theta_0=0.1451$. Then, with $\alpha=-0.05$ in Fig. 3. (a) shows that the origin is asymptotically stable. The Neimark-Sacker bifurcation occurs when $\alpha=0$. As consequence of this the origin loses its stability. When $\alpha=0.05$ the origin is unstable and a stable quasi-periodic orbit appears.



(a)



(b)

Fig. 3. The quasi-periodic orbits with the following parameters; θ_0=0.1451. (a): α=-0.05 ; (b): α=0.05, $a(0)$=-1.

5 Study of Arnold tongue

Near the Neimark-Sacker bifurcations, it can be observed the phenomenon of phase locking [12]. It is characterized by the transformation of the quasi-periodic orbits originated after the bifurcation into periodic orbits. In general, we can associate to a periodic or quasi-periodic orbit a number, called the rotation number. Its definition is

$$\rho = \frac{1}{2\pi} Lim_{k\to\infty} \frac{a(\varphi) + a(P(\varphi)) + \cdots + a(P^{k-1}(\varphi))}{k} \quad (23)$$

where

$$a(\varphi) = P(\varphi) - \varphi$$
$$P(\varphi) = \varphi + 2\pi\upsilon \,(\text{mod } 2\pi)$$

This number is estimated considering the rotation around the origin

$$P^k(\varphi) = \arctan g\left(\frac{y(k)}{x(k)}\right) \quad (24)$$

When the eigenvalues are on the unit circle and their phase can be expressed as a rational number (say q/p) multiplied by 2π the orbit is periodic. In general, this number can be irrational and this results in a quasi-periodic orbit. The rotation number estimates this quantity and it indicates that in p mapping iterations the state completes q revolutions if it is rational. Next to the unit circle there can be regions where this number is constant and rational, surrounded by zones in which is irrational. These zones are called Arnold tongues [8], due to their characteristic shape, collapsing to the corresponding point of rational value on the unit circle.

We want to estimate the mains arnold tongue in the mapping equations (2.a) and (2.b). In the beginning it has got four parameter of the system, and we have fixed two parameter (w_{11}, w_{12}) in each Fig. 4, and change the weight determinant and the diagonal element of the weight matrix sum.

In the fig. 4 it can show the preponderance of the period-four orbits (1/4 number rotation) in front of to the rest of period orbit. In other hand, it can extract to the figure that the probability of the quasi-periodic orbit present is similar to the period orbits ones. Also, in the intersection between different arnold tongues, it can find experimentally the coexistence of the different cycles periodic orbits, in this case there are several rotation numbers depend on the initial point used to calculate them.

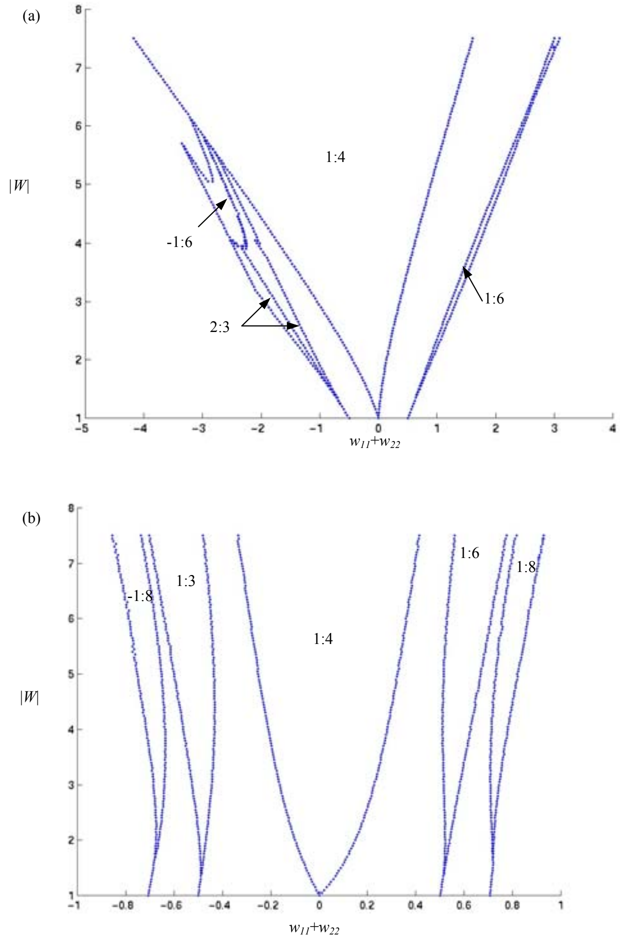


Fig. 4. The Arnold Tongues main representation with its rotation number. a) w_{11}= 0.217 y w_{12}=0.050 b) w_{11}=0.100 y w_{12}=10.000.

6 Conclusion

In this paper a simple discrete recurrent two-neuron network model has been considered. We have discussed the fixed points stability. We have shown the quasi-periodic orbits associated with the Neimark-Sacker bifurcation. In the limit, this orbit describes closed invariant trajectories. We also studied also their stability, and found them stable by applying normal form theory at their bifurcation. In addition to this, the necessary condition to produce the Neimark-Sacker bifurcation has been stated. We show the phase-locking phenomena when the tangent hyperbolic function show the Arnold Tongue function where appear the periodic orbit with different periodic.

The two-neuron networks discussed above are quite simple, but they are potentially useful since the complexity found in these simple cases might be carried over to larger discrete recurrent neural networks. There exists the possibility of generalising some of these results to higher dimensions.

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