On the effective heat conduction properties of heterogeneous media with multiple scales

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Abstract: - We study the asymptotic behaviour of generalized heat equations describing a periodic heterogeneous material with multiple scales when the fineness of the structure goes to zero. For different ratio between the characteristic sizes of the two spatial scales and the single time scale we find different equations, defined on a representative unit, providing us with the connection between the microstructure and the effective properties.

Key-Words: - Heterogeneous material, *G*-convergence, Homogenization, Multiple scales, Effective properties, Monotone operators

1 Introduction

We will study the generalized heat equation

$$\partial_t u^\varepsilon(x,t) - \nabla\cdot a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^r},\nabla u^\varepsilon\right) = f(x,t) \text{ in } \Omega\times(0,T),$$
$$u^\varepsilon(x,0) = u_0(x) \text{ in } \Omega,$$
$$u^\varepsilon(x,t) = 0 \text{ on } \partial\Omega\times(0,T), \qquad (1)$$

where the function *a* oscillates heavily for small ε. The bounded open set Ω represents a piece of some heterogeneous material and $(0,T)$ is the time interval to be studied. Solving this equation numerically for a fixed small ε may be difficult or even impossible. It is often enough to find the effective properties, that is, how the material behaves on a macroscopic level, and the associated solution. For this purpose we use homogenization, see e.g. [2] and [3].

The idea is to study a sequence of equations corresponding to an increasing fineness of the structure and see if there will be a stabilization of the properties. More precisely, the homogenization problem consist in studying the asymptotic behavior of the corresponding sequence of solutions u^ε and finding the limit equation which admits the limit u as its unique solution. This equation is called the homogenized problem and the solution is an approximation of the solution to (1) for small ε. The effective properties that appear in this equation can be attained from certain equations defined on representative units. This means that the numerical calculations simplifies considerably in the sense that there are no rapid oscillations in these problems.

2 *G*-convergence

The problem posed in (1) in the introduction means that we want to investigate the convergence of sequences of operators in certain evolution problems. A sequence of operators A^h can be said to converge to a limit operator A in the sense that a sequence of solutions u^h to the corresponding problems

$$\partial_t u^h(x,t) - A^h u^h(x,t) = f(x,t) \text{ in } \Omega\times(0,T),$$
$$u^h(x,0) = u_0(x) \text{ in } \Omega,$$
$$u^h(x,t) = 0 \text{ on } \partial\Omega\times(0,T),$$

converges in some sense to the solution u to an equation

$$\partial_t u(x,t) - Au(x,t) = f(x,t) \text{ in } \Omega\times(0,T),$$
$$u(x,0) = u_0(x) \text{ in } \Omega,$$
$$u(x,t) = 0 \text{ on } \partial\Omega\times(0,T).$$

This kind of convergence is well studied under the name of parabolic *G*-convergence for operators of the type

$$A^h u = \nabla\cdot a^h(x,t,\nabla u).$$

This means that the solutions u^h to the sequence of problems

$$\partial_t u^h(x,t) - \nabla\cdot a^h(x,t,\nabla u) = f(x,t) \text{ in } \Omega\times(0,T),$$
$$u^h(x,0) = u_0(x) \text{ in } \Omega,$$
$$u^h(x,t) = 0 \text{ on } \partial\Omega\times(0,T), \qquad (2)$$

converges to the solution u to

$$\partial_t u(x,t) - \nabla\cdot b(x,t,\nabla u) = f(x,t) \text{ in } \Omega\times(0,T),$$
$$u(x,0) = u_0(x) \text{ in } \Omega,$$
$$u(x,t) = 0 \text{ on } \partial\Omega\times(0,T),$$

where a^h and b fulfil certain monotonicity and growth conditions, see [7], [8] and [9]. In the next section we will study a special case of this type of convergence.

3 Homogenization

G-convergence guaranties the existence of the limit equation, but does not tell us much about the operator $b(x,t,\cdot)$ representing the effective heat conduction properties. We will study the equation (1), which means that we have chosen

$$a^h(x,t,\cdot) = a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^r},\cdot\right)$$

For the special case when the operator is linear, i.e.,

$$a^h(x,t,\nabla u) = a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^r}\right)\nabla u(x,t),$$

we have the usual linear heat equation for a material whose properties varies on two spatial scales and one time scale, see [5]. This is illustrated in the figures 1 and 2 below, for *t* fixed and for $\varepsilon = 0.6$ and $\varepsilon = 0.1$ respectively.

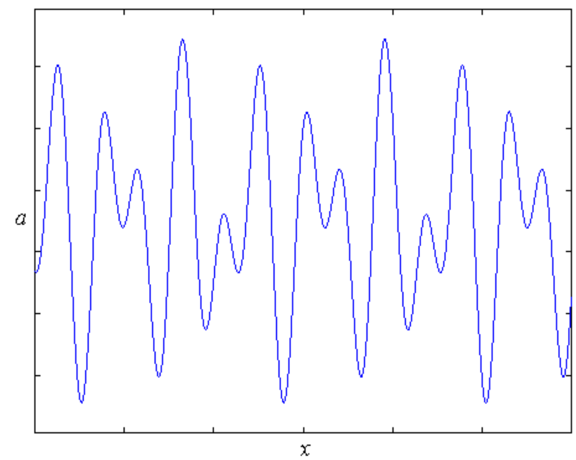

Fig.1: The pattern of oscillations of *a* for $\varepsilon = 0.6$.

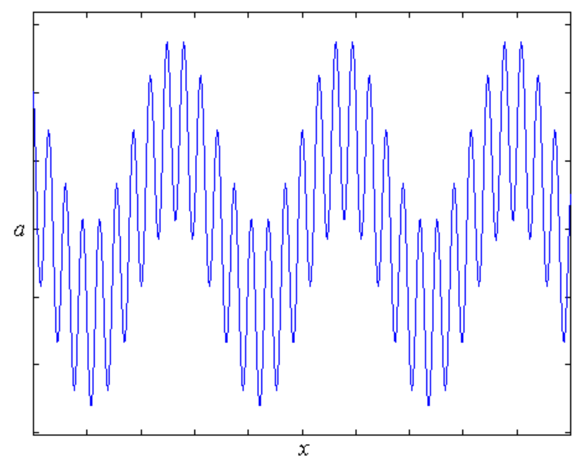

Fig.2: The pattern of oscillations of *a* for $\varepsilon = 0.1$.

We will see that it is possible to determine the limit operator and discover that this is done in different ways for different values of r.

For $r = 2$ the sequence of operators a^h corresponding to the sequence of problems (1)

G-converges to a limit b, that is $\{u^{\varepsilon}\}$ converges to u which is the unique solution to the homogenized problem

$$\begin{aligned}\partial_t u(x,t) - \nabla \cdot b(x,t,\nabla u) &= f(x,t) \text{ in } \Omega\times(0,T),\\ u(x,0) &= u_0(x) \text{ in } \Omega,\\ u(x,t) &= 0 \text{ on } \partial\Omega.\end{aligned}$$

It turns out that the limit operator b takes the form

$$b(x,t,\nabla u) = \int_0^1 \int_{Y_1} \int_{Y_2} a\left(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) \, dy_1 dy_2 ds,$$

where u_1 and u_2 can be attained from the system of local problems

$$\begin{aligned}-\partial_s u_1(x,t,y_1,y_2,s) + \nabla_{y_1} \cdot a\left(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) &= 0,\\ \nabla_{y_2} \cdot a\left(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) &= 0,\end{aligned}$$

solved for (y_1, s) in $Y_1 \times (0,1)$ and (y_2, s) in $Y_2 \times (0,1)$ respectively, assuming u_1 and u_2 to be periodic in these variables with respect to the interval (0,1) and the unit cubes Y_1 and Y_2. The proofs are found in [4] and [5], where we use multi-scale convergence, see Section 5, and perturbed testfunctions, see [4].

For $2 < r < 3$ the homogenized problem and the limit operator can be shown to have the same form as in the previous case. However, the equations needed to translate the properties on the microscopic level to global effective properties are different, namely

$$\begin{aligned}\nabla_{y_2} \cdot a\left(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) &= 0,\\ \partial_s u_1(x,t,y_1,s) &= 0,\\ \nabla_{y_1} \cdot a\left(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) &= 0.\end{aligned}$$

See [4] and [5]. Other cases when $r > 0$ can be investigated using similar methods. Numerical experiments for this kind of problems are being prepared.

4 Conclusions

Compared to [7], [8] and [9] we study a problem containing an additional spatial scale. The consequences of the relation between the speed of the oscillations in these scales on the one hand and in the time scale on the other is investigated. We see that different choices of oscillation frequencies result in different local problems, which capture the oscillations on the microscopic level. This information is then used to describe the properties on a macroscopic level. For the briefness and lucidity of the paper we consider the problem (1) only for the cases $r = 2$ and $2 < r < 3$. For $r = 2$ we have the correspondence of the self-similar homogenization case in [7] and for $2 < r < 3$ one of the cases of non self-similar homogenization. The other cases can be treated by similar extensions of the results in [5]. The new achievement compared to [5] is that we allow the response of the material to be nonlinear under certain monotonicity and boundedness conditions.

5 Mathematical background

An efficient method for solving homogenization problems is based on two-scale convergence which was introduced by Nguetseng, see [6], in the late 80's. He proved that under the assumption that

$$\int_\Omega u^{\varepsilon}(x)^2 \, dx < C,$$

it holds that

$$\lim_{\varepsilon\to 0} \int_\Omega u^{\varepsilon}(x) v\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u(x,y) v(x,y) \, dxdy$$

for all $v(x, y)$ which are smooth and periodic with respect to the unit cube Y in the second argument. The sequence $\{u^{\varepsilon}\}$ is said to two-scale converge to u. The corresponding convergence concept with multiple scales is studied by Allaire and Briane in [1].

For the treatment of our problem we use a type of convergence which includes 3 spatial scales and 2 time scales. A sequence $\{u^{\varepsilon}\}$ is said to 3,2-scale converge to u if

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u^\varepsilon(x,t) v\left(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^r}\right) dxdt =$$

$$\int_0^T \int_\Omega \int_0^1 \int_{Y_1} \int_{Y_2} u(x,t,y_1,y_2,s) v(x,t,y_1,y_2,s)\ dxdtdy_1dy_2ds$$

for all v periodic in y_1, y_2 and s. This is denoted

$$u \quad x,t \quad {}^{3,2} \quad u\ x,t,y_1,y_2,s\ .$$

Under certain boundedness conditions on u^ε, its gradient and its time derivative, which follow naturally from (1), it holds that

$$\boxed{u \quad x,t \quad {}^{3,2} \quad u\ x,t \ . \quad {}_{y_1}u_1\ x,t,y_1,s\ . \quad {}_{y_2}u_2\ x,t,y_1,y_2,s\ ,}$$

where u_1 and u_2 are periodic in y_1, s and y_1, y_2, s respectively. The functions u_1 and u_2 are the same as those appearing in the local problems, see Section 3. For details see [1], [4] and [5]. In the proof of the results in the preceding section the key to the general monotone case is to prove that

$$\boxed{a_j\left(\frac{x}{\ }, \frac{x}{\ {}^2}, \frac{t}{\ {}^r}, \quad u \right) {}^{3,2} \quad a_j\ y_1,y_2,s,\ u. \quad {}_{y_1}u_1. \quad {}_{y_2}u_2\ .}$$

This is accomplished by means of perturbed testfunctions in [4].

The convergence of the sequences $\{u^\varepsilon\}$ and $\{u^h\}$ in (1) and (2) respectively takes place in the norm topology of $L^2(\Omega \times (0,T))$ and in the weak topology of $W_2^1\left(0,T;W_0^{1,2}(\Omega),L^2(\Omega)\right)$, see [10].

The operators $\nabla \cdot a^h(x,t,\nabla u)$ are monotone in the sense that

$$\int_0^T \int_\Omega \left(a^h(x,t,\nabla u) - a^h(x,t,\nabla v)\right)(\nabla u - \nabla v)\ dxdt \geq$$

$$C \int_0^T \int_\Omega |\nabla u - \nabla v|^2\ dxdt,$$

see [4] and [11].

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