Transient Radiation Transfer in Finite Media using Pomraning-Eddington Approximation

S. A. El-Wakil , A. R. Degheidy, and M. Sallah*
Theoretical Physics Research Group, Physics Department, Faculty of Science, Mansoura University, Mansoura P. O. Box. 35516, Egypt

ABSTRACT: The equation of transient radiation transfer with linear anisotropic scattering in finite plane-parallel media is studied. Pomraning-Eddington approximation is used to solve the problem. Numerical results for reflectivity and transmissivity at different times are done for an isotropic medium which is assumed to have specular-reflecting boundaries. Weight functions are introduced to force the boundary conditions to be fulfilled.

Keywords: Radiation Transfer, Transient, Finite Media

1 Introduction

The studying and solving the time-dependent transport problems have a very wide interesting applications such as in reactor physics and astrophysics. The propagation of a laser pulse through a scattering and absorbing medium is of importance in many fields, including lidar and optical communication [1], remote sensing [2], computer tomography [3] and photodynamic therapy [4]. One of the very interesting applications of the time-dependent transport equation is the heat transfer. Heat transfer in dielectric materials and semiconductors is predominantly by phonons, which are the quanta of crystal vibrational energy as the photons is the quanta of the radiation. Transient heat transport by phonons is of vital importance in several technological applications. For example, microelectronics devices are composed mainly of thin films of semiconductors such as Si or $GaAs$, of dielectrics such as SiO_2 and Si_3N_4 ,and of metals [5].

Several approaches have been proposed to solve the one-dimensional time-dependent transport equation in finite slabs, such as, the multiple-collision [6], discrete ordinates and semi-analytical numerical [7] methods.

In this work, we are focusing our attention to solve the monoenergetic time-dependent radiation transfer equation in a finite slab medium. The medium is considered to have specular-reflecting boundaries, and an externally-incident flux which assumed to be angular-dependent. The time-dependent problem is transformed into a stationary-like problem, and then Pomraning-Eddington approximation is used to solve it. After obtaining the radiation intensity, we can compute some physical quantities such as reflectivity and transmissivity. A weight function is introduced to force the assumed specular-reflecting boundary conditions to be fulfilled. For the sake of comparison, two different weight functions are used.

*E-mail address: msallahd@yahoo.com

2 Analysis

The time-dependent, monoenergetic, radiation transfer equation with anisotropic scattering takes the form [8]

$$\left[\frac{1}{\nu}\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial z}+\sigma(t,z)\right]N(t,z,\mu)=$$
$$\frac{\sigma_s(t,z)}{2}\int_{-1}^{1}P(\mu,\acute{\mu})N(t,z,\acute{\mu})d\acute{\mu}+Q(t,z)$$
$$0\le z\le b\ ,\ -1\le\mu\le 1 \qquad (1)$$

where
$N(t,z,\mu)$ is the radiation intensity with temporal variable t, geometrical space variable z, and the angular variable μ, ν is the radiation speed, $\sigma(z)$ is the total cross-section, $\sigma_s(z)$ is the scattering cross-section, and $Q(t,z)$ is the internal energy source. $P(\mu,\acute{\mu})$ is the anisotropic scattering phase function which is given by [9]

$$P(\mu,\acute{\mu})=\sum_{n=0}^{\infty}a_nP_n(\mu)P_n(\acute{\mu}) \qquad (2)$$

where $P_n(\mu)$ is the Legendre polynomial functions, with $P_0(\mu)=1$, and $a_0=1$.

It is convenient to write Eq.(1) in terms of the optical depth space variable

$$x(z) = \int_0^z \sigma(z)dz \quad , \quad 0 \le x \le d \tag{3}$$

where the optical thickness d of the medium is

$$d(b) = \int_0^b \sigma(z)dz \tag{4}$$

In terms of x Eq.(1) becomes

$$\left[\frac{1}{u}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} + 1\right] I(t,x,\mu) =$$

$$\frac{\omega}{2}\int_{-1}^{1} P(\mu,\acute{\mu}) I(t,x,\acute{\mu})d\acute{\mu} + S(t,x) \tag{5}$$

where

$$\begin{aligned} I(t,x,\mu) &\equiv N(t,z,\mu)\ , & (6.a)\\ u &= \nu\sigma\ ,\quad \omega = \sigma_s/\sigma & (6.b)\\ S(t,x) &= Q(t,z)/\sigma & (6.c)\end{aligned}$$

Equation (5) is assumed to subject to boundary conditions

$$I(t,0,\mu) = \Gamma(\mu) + \rho_1^s I(t,0,-\mu) \quad , t=0 \tag{7.a}$$

$$I(t,d,-\mu) = \rho_2^s I(t,d,\mu) \qquad , t>0 \tag{7.b}$$

where $\Gamma(\mu)$ is the angular-dependent externally-incident flux on the left boundary, and ρ_i^s is the specular reflectivity of the boundaries, $(i = 1, 2)$.

For solving equation (5), we use the transformation [10]

$$\zeta = x + ut \tag{8}$$

to have

$$\left[(1+\mu)\frac{\partial}{\partial \zeta} + 1\right] I(\zeta,\mu) = \frac{\omega}{2}\int_{-1}^{1} P(\mu,\acute{\mu}) I(\zeta,\acute{\mu})d\acute{\mu} + S(\zeta) \tag{9}$$

For linear approximation of the anisotropic scattering phase function, Eq.(2) becomes

$$P(\mu,\acute{\mu}) = 1 + \bar{a}\mu\acute{\mu} \tag{10}$$

where $\bar{a}$ is the anisotropy parameter which is given in terms of Legendre polynomial coefficients [9]

Considering the source-free problem and using Eq.(10) in Eq.(9), we have

$$\left[(1+\mu)\frac{\partial}{\partial \zeta} + 1\right] I(\zeta,\mu) = \frac{\omega}{2}\left[E(\zeta) + \bar{a}\mu F(\zeta)\right] \tag{11}$$

where $E(\zeta)$ is the radiant energy, and $F(\zeta)$ is the net radiant heat flux, which are defined by

$$\begin{aligned} E(\zeta) &= \int_{-1}^{1} I(\zeta,\mu)d\mu & (12.a)\\ F(\zeta) &= \int_{-1}^{1} \mu I(\zeta,\mu)d\mu & (12.b)\end{aligned}$$

For solving this problem we use Pomraning-Eddington approximation, where the intensity assumed as[11]

$$I(\zeta,\mu) = \Sigma(\zeta,\mu)E(\zeta) + O(\zeta,\mu)F(\zeta) \tag{13}$$

where $\Sigma(\zeta,\mu)$ and $O(\zeta,\mu)$ are even and odd functions in μ and are slowly varying functions in ζ which are normalized as

$$\begin{aligned} \int_{-1}^{1} d\mu\Sigma(\zeta,\mu) &= 1 & (14)\\ \int_{-1}^{1} d\mu\mu O(\zeta,\mu) &= 1 & (15)\end{aligned}$$

Substituting Eq.(13) in Eq.(11), integrating over $\mu \in [-1,1]$, and using Eqs.(14, 15) one gets

$$\frac{dE(\zeta)}{d\zeta} + \frac{dF(\zeta)}{d\zeta} + \alpha E(\zeta) = 0 \tag{16}$$

In the same way, multiplying of Eq.(11) by μ and integrate over $\mu \in [-1,1]$ gives

$$D\,\frac{dE(\zeta)}{d\zeta} + \frac{dF(\zeta)}{d\zeta} + \beta F(\zeta) = 0 \tag{17}$$

where

$$\begin{aligned} \alpha &= 1-\omega\ ,\quad \beta = 1 - \frac{\bar{a}\omega}{3}\ , & (18)\\ D &= \int_{-1}^{1} \mu^2\Sigma(\mu)d\mu & (19)\end{aligned}$$

Eqs.(16) and (17) lead to the second-order differential equation satisfied by $E(\zeta)$

$$(1-D)\frac{d^2E(\zeta)}{d\zeta^2} + (\alpha+\beta)\frac{dE(\zeta)}{d\zeta} + \alpha\beta E(\zeta) = 0 \tag{20}$$

which has the solution

$$E(\zeta) = Ae^{-k^+\zeta} + Be^{-k^-\zeta} \tag{21}$$

Moreover,

$$F(\zeta) = \gamma^+ Ae^{-k^+\zeta} + \gamma^- Be^{-k^-\zeta} \tag{22}$$

where

$$2k^{\pm} = \frac{\alpha+\beta}{1-D} \mp \sqrt{\left[\frac{\alpha+\beta}{1-D}\right]^2 - \frac{4\alpha\beta}{1-D}} \quad (23)$$

and

$$\gamma^{\pm} = \frac{1}{\beta}\left[\alpha - (1-D)\, k^{\pm}\right] \quad (24)$$

The constants A & B are to be determined

Even and Odd functions can be obtained by substituting Eq.(13) in Eq.(11), separating the even and odd terms of the resultant equation, and then solve for the even and odd functions, we get

$$\Sigma(\mu) = \left(\frac{\omega}{2}\right) \frac{D\,(1-r) + \overline{a}D\,(\alpha - R)\,\mu^2}{D\,(1-R)\,(1-r) - (\alpha-R)\,(\beta-r)\,\mu^2} \quad (25)$$

and

$$O(\mu) = \left(\frac{\omega}{2}\right) \frac{(\beta - r + \overline{a}D\,(1-R))\,\mu}{D\,(1-R)\,(1-r) - (\alpha-R)\,(\beta-r)\,\mu^2} \quad (26)$$

where R is defined as [13]

$$R = -\frac{1}{E(\zeta)}\frac{dE(\zeta)}{d\zeta} \quad (27.a)$$

$$\text{and} \quad r = \beta\frac{\alpha - R}{\alpha - (1-D)\,R} \quad (27.b)$$

The two unknown parameters R and D can be determined by solving the two coupled transcendental equations which obtained by substituting Eq.(25) in Eqs.(14) and (19).

Finally, we can obtain the **solution** as

$$I(\zeta,\mu) = Ah_-(\mu)e^{-k^+\zeta} + Bh_+(\mu)e^{-k^-\zeta} \quad (30)$$

or

$$I(t,x,\mu) = Ah_-(\mu)e^{-k^+x}e^{-k^+ut} + Bh_+(\mu)e^{-k^-x}e^{-k^-ut} \quad (31)$$

with

$$h_{\pm}(\mu) = \frac{\frac{1-r}{1-R} + \gamma^{\pm}\left[\overline{a} + \frac{\beta-r}{D(1-R)}\right]\mu + \overline{a}\left[\frac{\alpha-R}{1-R}\right]\mu^2}{1 - r - \frac{(\alpha-R)(\beta-r)}{D(1-R)}\mu^2} \quad (32)$$

To determine the constants A and B , a weight function $W(\mu)$ is introduced in order to force the boundary conditions to be fulfilled, as

$$\int_0^1 d\mu W(\mu)\left[I(t,0,\mu) - \rho_1^s I(0,-\mu)\,\right] = I_0 \quad (33.a)$$

$$\int_0^1 d\mu W(\mu)\left[\; I(t,d,-\mu) - \rho_2^s I(t,d,\mu)\;\right] = 0 \quad (33.b)$$

Using Eq.(31) we obtain

$$A = \frac{-I_0\left(I_-^- - \rho_2^s I_-^+\right)}{Q} e^{-\eta(d+ut)} \quad , \quad (34)$$

$$B = \frac{I_0\left(I_+^- - \rho_2^s I_+^+\right)}{Q} \quad (35)$$

where

$$Q = \left(I_-^+ - \rho_1^s I_-^-\right)\left(I_+^- - \rho_2^s I_+^+\right) - \left(I_+^+ - \rho_1^s I_+^-\right)\left(I_-^- - \rho_2^s I_-^+\right)e^{-\eta(d+ut)} \quad (36.a)$$

$$\eta = k^- - k^+ \quad , \quad (36.b)$$

$$I_{\pm}^{\mp} = \int_0^1 d\mu W(\mu)\; h_{\pm}(\mp\mu) \quad , \quad (36.c)$$

$$I_0 = \int_0^1 d\mu W(\mu)\Gamma(\mu) \quad (36.d)$$

3 Numerical Results and Calculations

In this section numerical calculations are done to calculate the reflectivity R_r and transmissivity T_r for a slab of thickness d, which are defined as

$$R_r = \int_0^1 d\mu\mu I(t,0,-\mu) \quad \text{at} \;\; t=0, \;\; \mu>0 \quad (37)$$

$$T_r = \int_0^1 d\mu\mu I(t,d,\mu) \quad \text{at} \;\; t\geq 0, \;\; \mu>0 \quad (38)$$

Using Eq.(31) in Eqs.(37) and (38) one gets

$$R_r = AJ_1^r + BJ_2^r \quad (39)$$

and

$$T_r = AJ_1^t e^{-k^+(d+ut)} + BJ_2^t e^{-k^-(d+ut)} \quad (40)$$

where

$$J_1^r = \int_0^1 d\mu\mu\; h_+(-\mu), \;\; J_2^r = \int_0^1 d\mu\mu\; h_-(-\mu) \quad (41)$$

$$J_1^t = \int_0^1 d\mu\mu\; h_+(\mu), \;\; J_2^t = \int_0^1 d\mu\mu\; h_-(\mu) \quad (42)$$

The angular-dependent externally-incident flux $\Gamma(\mu)$ is assumed to have the form

$$\Gamma(\mu) = \mu^l \;, \qquad l = 0,1,2,\cdots\cdots \quad (43)$$

Two different weight functions are suggested to do the calculations, namely [11,12]

$$W_1(\mu) = \mu \quad (44)$$

$$W_2(\mu) = \frac{\sqrt{3}}{2}\mu\left(1+\frac{3}{2}\mu\right) \quad (45)$$

Table (1) shows the data of the reflectivity R_r for four groups of l and ρ_2^s, with varying the thickness of the medium d at different instants t . Table (2) gives the results for transmissivity T_r for the same parameters as in table(1). Table(3) gives the results of reflectivity R_r and transmissivity T_r for $l = 1$ and $\rho_2^s = 0.5$ with varying the time t for different values of the thickness d.

4 Conclusion

The reflectivity and transmissivity at the boundaries of a finite slab medium are calculated for the time-dependent monoenergetic radiation transfer equation with linear anisotropic scattering. The medium is considered to have specular-reflecting boundaries, and the externally-incident flux is assumed to be angular-dependent. A weight function is introduced to force the assumed specular-reflecting boundary conditions to be fulfilled. Because of the lake of the corresponding data in the literatures we use two different weight functions for the sake of comparison which give good agreement with each other.

On the other hand, the above analysis of the time dependent radiation transfer gives a good description for the transient heat transfer, where the transient heat transfer is described by the equation of phonon radiative transfer (EPRT) [5]

$$\left[\frac{1}{v}\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}+\frac{1}{v\tau_R}\right] I_\omega\left(x,\mu,t\right)=\frac{1}{2v\tau_R}\int_{-1}^{1} I_\omega\left(x,\mu,t\right)d\mu \quad (46)$$

which is a simple form of radiative transfer equation with isotropic scattering and the cross-section $\sigma_t = \frac{1}{v\tau_R}$, where $v\tau_R$ is the effective mean free and τ_R is the relaxation time.

In future work we will use this analysis to calculate the temperature of a finite thin medium at different times.

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Table(1) The reflectivity R_r for a medium of transparent left boundary for $\omega = 0.9$

(l, ρ_2^s)	$(0,0)$		$(1,0)$		$(2,0.25)$		$(2,0.5)$	
d	$W_1(\mu)$	$W_2(\mu)$	$W_1(\mu)$	$W_2(\mu)$	$W_1(\mu)$	$W_2(\mu)$	$W_1(\mu)$	$W_2(\mu)$
				$t = 0.0$				
0.01	0.44746	0.43342	0.29830	0.30701	0.22171	0.23805	0.21968	0.23771
0.10	0.44897	0.43364	0.29931	0.30716	0.22229	0.23814	0.22006	0.23777
1.00	0.46875	0.43646	0.31250	0.30916	0.23095	0.23951	0.22648	0.23880
3.00	0.49695	0.44027	0.33130	0.31186	0.24775	0.24204	0.24615	0.24180
5.00	0.49983	0.44064	0.33322	0.31212	0.24987	0.24235	0.24977	0.24233
				$t = 0.1$				
0.01	0.44914	0.43367	0.29943	0.30718	0.22236	0.23815	0.22010	0.23778
0.10	0.45077	0.43391	0.30051	0.30735	0.22300	0.23826	0.22052	0.23785
1.00	0.47115	0.43679	0.31410	0.30939	0.23215	0.23970	0.22751	0.23896
3.00	0.49735	0.44032	0.33157	0.31189	0.24804	0.24208	0.24662	0.24187
5.00	0.49985	0.44064	0.33323	0.31212	0.24989	0.24235	0.24980	0.24233
				$t = 3.0$				
0.01	0.49699	0.44027	0.33133	0.31186	0.24778	0.24204	0.24620	0.24181
0.10	0.49735	0.44032	0.33157	0.31189	0.24804	0.24208	0.24662	0.24187
1.00	0.49926	0.44057	0.33284	0.31207	0.24945	0.24228	0.24902	0.24222
3.00	0.49996	0.44065	0.33331	0.31213	0.24997	0.24236	0.24995	0.24236
5.00	0.49999	0.44066	0.33333	0.31214	0.25000	0.24237	0.25000	0.24237

Table(2) The transmissivity T_r for a medium of transparent left boundary for $\omega = 0.9$

(l, ρ_2^s)	$(0,0)$		$(1,0)$		$(2,0.25)$		$(2,0.5)$	
d	$W_1(\mu)$	$W_2(\mu)$	$W_1(\mu)$	$W_2(\mu)$	$W_1(\mu)$	$W_2(\mu)$	$W_1(\mu)$	$W_2(\mu)$
				$t = 0.0$				
0.01	0.49799	0.48295	0.33199	0.34209	0.24922	0.26789	0.24944	0.27016
0.10	0.47951	0.46420	0.31967	0.32881	0.24197	0.25978	0.24422	0.26434
1.00	0.26915	0.25463	0.17943	0.18036	0.15248	0.16085	0.17588	0.18842
3.00	0.02165	0.01984	0.01443	0.01405	0.01486	0.01516	0.02371	0.02462
5.00	0.00101	0.00093	0.00068	0.00066	0.00071	0.00072	0.00119	0.00122
				$t = 0.1$				
0.01	0.47741	0.46208	0.31827	0.32730	0.24114	0.25886	0.24362	0.26368
0.10	0.45812	0.44255	0.30541	0.31348	0.23350	0.25032	0.23811	0.25754
1.00	0.24608	0.23216	0.16405	0.16445	0.14151	0.14892	0.16652	0.17810
3.00	0.01865	0.01708	0.01243	0.01210	0.01284	0.01309	0.02062	0.02140
5.00	0.00087	0.00079	0.00058	0.00056	0.00061	0.00062	0.00102	0.00105
				$t = 3.0$				
0.01	0.02133	0.01954	0.01422	0.01384	0.01465	0.01495	0.02338	0.02428
0.10	0.01865	0.01708	0.01243	0.01210	0.01284	0.01309	0.02062	0.02140
1.00	0.00475	0.00434	0.00316	0.00307	0.00332	0.00337	0.00550	0.00569
3.00	0.00022	0.00020	0.00014	0.00014	0.00015	0.00015	0.00025	0.00026
5.00	9×10^{-6}	9×10^{-6}	6×10^{-6}	6×10^{-6}	7×10^{-6}	7×10^{-6}	1×10^{-5}	1×10^{-5}

Table(3) The reflectivity R_r and The transmissivity T_r for isotropic scattering with $l = 1$ and $\rho_2^s = 0.5$

t	$\mathbf{R}_r$ $W_1(\mu)$	$\mathbf{R}_r$ $W_2(\mu)$	$\mathbf{T}_r$ $W_1(\mu)$	$\mathbf{T}_r$ $W_2(\mu)$	$\mathbf{R}_r$ $W_1(\mu)$	$\mathbf{R}_r$ $W_2(\mu)$	$\mathbf{T}_r$ $W_1(\mu)$	$\mathbf{T}_r$ $W_2(\mu)$
	$\omega = 0.8$				$\omega = 0.95$			
				$d = 0.01$				
0.0	0.24998	0.26103	0.33246	0.34782	0.31350	0.31363	0.33264	0.34801
0.1	0.25067	0.26144	0.32363	0.33853	0.31383	0.32386	0.32542	0.34014
0.5	0.25445	0.26366	0.28669	0.29960	0.31559	0.32625	0.29034	0.30200
1.0	0.26225	0.26822	0.23543	0.24554	0.31880	0.32757	0.23207	0.23929
5.0	0.33214	0.30834	0.00188	0.00193	0.33322	0.32782	0.00149	0.00148
				$d = 0.1$				
0.0	0.25060	0.26140	0.32452	0.33946	0.31380	0.31363	0.32617	0.34096
0.1	0.25137	0.26185	0.31557	0.33003	0.31417	0.32336	0.31842	0.33251
0.5	0.25558	0.26432	0.27792	0.29035	0.31608	0.32587	0.28099	0.29188
1.0	0.26409	0.26930	0.22554	0.23511	0.31948	0.32732	0.22021	0.22663
5.0	0.33228	0.30842	0.00163	0.00167	0.33323	0.32760	0.00130	0.00129
				$d = 1.0$				
0.0	0.26205	0.26811	0.23652	0.24669	0.31872	0.31358	0.23337	0.24067
0.1	0.26409	0.26930	0.22554	0.23511	0.31948	0.31825	0.22021	0.22663
0.5	0.27391	0.27502	0.17968	0.18685	0.32271	0.32101	0.16565	0.16899
1.0	0.28898	0.28373	0.12222	0.12661	0.32656	0.32336	0.10293	0.10392
5.0	0.33304	0.30885	0.00039	0.00039	0.33331	0.32391	0.00033	0.00033
				$d = 3.0$				
0.0	0.31611	0.29927	0.03940	0.04054	0.33131	0.31355	0.02925	0.02916
0.1	0.31801	0.30034	0.03442	0.03540	0.33157	0.31389	0.02543	0.02534
0.5	0.32397	0.30373	0.01953	0.02005	0.33232	0.31424	0.01432	0.01424
1.0	0.32848	0.30628	0.00921	0.00945	0.33284	0.31476	0.00684	0.00679
5.0	0.33332	0.30901	0.00002	0.00002	0.33333	0.31494	0.00002	0.00002
				$d = 5.0$				
0.0	0.33212	0.30833	0.00191	0.00196	0.33322	0.31355	0.00151	0.00150
0.1	0.33228	0.30842	0.00163	0.00167	0.33323	0.31357	0.00130	0.00129
0.5	0.33274	0.30868	0.00086	0.00088	0.33328	0.31359	0.00071	0.00070
1.0	0.33304	0.30885	0.00039	0.00039	0.33331	0.31362	0.00033	0.00033
5.0	0.33333	0.30902	6×10^{-7}	6×10^{-7}	0.33333	0.31363	7×10^{-7}	7×10^{-7}