The Universal E - Subcompactification of Semigroups

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Abstract: In this article we consider the enveloping semigroup of a flow generated by the action of a semitopological semigroup on any of its semigroup compactifications, and we define the notion of E-compactification and E- subcompactification and also we get the universal E- subcompactification.

Key-Words :- Semigroup, reductive semigroup, semigroup compactification, enveloping semigroup, E-subcompactification.

1 Introduction

A semigroup S is called *right reductive* if for each $a, b \in S$, from at=bt for every $t \in S$, it follows that a=b. For example, all right cancellative semigroups and semigroups with a right identity, are right reductive. Throughout this article S will be a semitopological semigroup. By a *semigroup compactification* of S we mean a pair (ψ, X), where X is a compact Hausdorff right topological semigroup, and $\psi : S \longrightarrow X$ is a continuous homomorphism with dense image such that, for each $s \in S$, the mapping $x \longrightarrow \psi(s)x : X \longrightarrow X$ is continuous. The C^*-algebra of all bounded complex-valued continuous functions on S, will be denoted by $\mathcal{C}(S)$. For $\mathcal{C}(S)$ the left and right translations, L_s and R_t, are defined for each $s, t \in S$ by $(L_s f)(t) = f(st) = (R_t f)(s), f \in \mathcal{C}(S)$.

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The subset $\mathcal{F}$ of $\mathcal{C}(S)$ is said to be left translation invariant, if for all $s \in S, L_s\mathcal{F} \subseteq \mathcal{F}$. A left translation invariant unital C^*-subalgebra $\mathcal{F}$ of $\mathcal{C}(S)$ is called *m-admissible* if the function $s \longrightarrow T_\mu f(s) = \mu(L_s f)$ is in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$:=the spectrum of $\mathcal{F}$. Then the product of $\mu, \nu \in S^{\mathcal{F}}$ can be defined by $\mu\nu = \mu \circ T_\nu$ and the Gelfand topology on $S^{\mathcal{F}}$ makes $(\epsilon, S^{\mathcal{F}})$ a semigroup compactification (called the $\mathcal{F}$-compactification) of S, where $\epsilon : S \longrightarrow S^{\mathcal{F}}$ is the evaluation mapping.

Some m-admissible subalgebras of $\mathcal{C}(S)$ are: $\mathcal{LMC}$:= left multiplicatively continuous functions, $\mathcal{D}$:=distal functions, $\mathcal{MD}$:=minimal distal functions, and $\mathcal{SD}$:=strongly distal functions. We also write $\mathcal{GP}$ for $\mathcal{MD} \cap \mathcal{SD}$; and we define $\mathcal{LZ} := \{f \in \mathcal{C}(S); f(st) = f(s) \text{ for all } s, t \in S\}$.
For a discussion of the universal property of the corresponding compactifications of these function algebras see [1] and also [4].
Let (ψ, X) be a compactification of S, then the mapping $\sigma : S \times X \longrightarrow X$, defined by $\sigma(s, x) =$

$\psi(s)x$, is separately continuous and so (S,X,σ) is a flow. If Σ_X denotes the enveloping semigroup of the flow (S,X,σ) (i.e., the pointwise closure of semigroup $\{\sigma(s,.) : s\in S\}$ in X^X) and the mapping $\sigma_X : S\longrightarrow\Sigma_X$ defined by $\sigma_X(s)=\sigma(s,.)$ for all $s\in S$, then (σ_X,Σ_X) is a compactification of S (see [1;1.6.5]).

One can easily verify that $\Sigma_X=\{\lambda_x : x\in X\}$, where $\lambda_x(y)=xy$ for each $y\in X$. If we define the mapping $\theta : X\longrightarrow\Sigma_X$ by $\theta(x)=\lambda_x$, then θ is a continuous homomorphism with the property that $\theta\circ\psi=\sigma_X$. So (σ_X,Σ_X) is a factor of (ψ,X), that is $(\psi,X)\geq(\sigma_X,\Sigma_X)$. By definition, θ is one-to-one, if and only if X is right reductive. So we get the next proposition, which is an extension of the Lawson's result [5; 2.4(ii)]:

1.1.Proposition. Let (ψ,X) be a compactification of S. Then $(\sigma_X,\Sigma_X)\cong(\psi,X)$, if and only if X is right reductive.

A compactification (ψ,X) is called *reductive*, if X is right reductive. For example, the $\mathcal{MD}$, $\mathcal{GP}$ and $\mathcal{LZ}$-compactifications, are reductive. A compactification (ψ,X) is called E- *compactification* if $(\psi,X)\cong(\sigma_Y,\Sigma_Y)$ for some compactification (θ,Y) of S. Clearly every reductive compactification is E- compactification but the converse is not, in general true. For example see [2; 2.2].

It is easy to see that $(\sigma_{S^{\mathcal{LMC}}},\Sigma_{S^{\mathcal{LMC}}})$ is the universal E- compactification of S.

2 E- Subcompactification

2.1.Definition. Let (ψ,X) be a compactification of S. We say that (θ,Y) is an E- subcompactification of (ψ,X) if $(\sigma_Y,\Sigma_Y)\leq(\psi,X)$.

By the above, every compactification is also E-subcompactification of itself. Now we are going to construct the universal E- subcompactification of S. For this end we need the following lemma.

2.2.Lemma. Let (ψ,X) be the subdirect product of the family $\{(\psi_i,X_i) : i\in I\}$ of compactifications of S. Then (σ_X,Σ_X) is isomorphic to the subdirect product of the family $\{(\sigma_{X_i},\Sigma_{X_i}) : i\in I\}$ (i.e., $\vee(\sigma_{X_i},\Sigma_{X_i})\cong(\sigma_X,\Sigma_X)$).

proof: By [1; 3.2.5]), for each $i\in I$, there exists a homomorphism p_i of (ψ,X) onto (ψ_i,X_i) . So, by [1; 1.6.7], for each $i\in I$, there exists a unique continuous homomorphism π_i of (σ_X,Σ_X) onto $(\sigma_{X_i},\Sigma_{X_i})$ such that

$$\pi_i(\zeta)(p_i(x))=p_i(\zeta(x)) \qquad (x\in X,\zeta\in\Sigma_X).$$

Suppose that $\zeta_1,\zeta_2\in\Sigma_X$. If $\pi_i(\zeta_1)=\pi_i(\zeta_2)$ for all $i\in I$, then

$$\begin{aligned} p_i(\zeta_1(x)) &= (\pi_i(\zeta_1))(p_i(x)) = (\pi_i(\zeta_2))(p_i(x)) \\ &= p_i(\zeta_2(x)) \end{aligned}$$

for all $x\in X$ and $i\in I$. Thus $\zeta_1=\zeta_2$. Therefore the family $\{\pi_i : i\in I\}$ separates the points of Σ_X. Hence the conclusion holds ([1; 3.2.5]).

2.3.Theorem. Every compactification (ψ,X) of S has the universal E- subcompactification.

proof: Let (ϕ,Y) be a compactification of S. Suppose $\{(\psi_i,X_i) : i\in I\}$ is a family of E-subcompactifications of (ϕ,Y), and (ψ,X) is the subdirect product of this family. We show that (ψ,X) is an E- subcompactification of (ϕ,Y), and so it is the universal E- subcompactification of (ϕ,Y). To see this, for each $i\in I$, we have $(\sigma_{X_i},\Sigma_{X_i})\leq(\phi,Y)$. So, by property of subdirect product, we have $\vee(\sigma_{X_i}, \Sigma_{X_i}) \leq (\phi, Y)$. By previous lemma $\vee(\sigma_{X_i}, \Sigma_{X_i}) \cong (\sigma_X, \Sigma_X)$. Hence $(\sigma_X, \Sigma_X) \leq (\phi, Y)$. This means that (ψ, X) is an E- subcompactification of (ϕ, Y).

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