Domain Decomposition Approach for Moving Interface Phenomenon by Using Boundary Element Method

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Abstract:-In this work, we present a new boundary element procedure for the density startified flow based on the domain decomposition method. The final system of equations for the whole region is obtained by adding the set of boundary integral equations of governing equation for each sub-domain in conjunction with compatibility and equilibrium conditions between their interfaces. Obtained results show the tendency to increase of iteration number in computation at complicated shape of the internal boundary.

Key-Words:-Interfacial Dynamics, Domain Decomposition Method, Boundary Element Scheme

1 Introduction

In many situations of hydrodynamic phenomenon, two-layer fluid is a dominant feature of fluid motion. Such two-layer fluid often contains jumps in the density across the interface of fluid domains. The interface of air and water or salt water and fresh water is obvious example [1]. The density jump may be assumed to occur in an infinitesimally thin interface in the mixture. Moreover, the behavior of such an interface is important to understand various hydrodynamic phenomena [2].

In the previous works, development of the efficient numerical procedure which is able to simulate the time evolution of an interface between two fluids with different densities are performed. The numerical solution procedure based on the sub-region boundary element method with the mixed Eulerian-Lagrangian approach developed in the moving boundary problems in potential field [3]. Now, cluster computation by work stations or personal computers becomes available in many laboratories all over the world. Because of their potential for both high-performance and cost-effectiveness, cluster computing will attract much more attention of researches, and they will take the most important part in engineering computation in stead of vector computing in near future. Under this situation, investigation of parallel Finite Element Method (FEM) algorithm [4, 5] based on the Domain Decomposition Method (DDM) is increasing. Recently, Kamiya et al. introduced DDM for the boundary element analysis in order to implement the parallel Boundary Element Method (BEM) computation [6, 7]. They showed the utility of BEM analysis with domain decomposition scheme for some potential and elastic problems. In this paper, we propose a new boundary element procedure for the density stratified flow based on the domain decomposition method. The final system of equations for the whole region is obtained by adding the set of boundary integral equations of governing equation for each sub-domain in conjunction with compatibility and equilibrium conditions between their interfaces. The present study is an attempt to develop parallel computation procedure for the interface motion of the two-layer fluid in a rectangular region based on domain decomposition method and two sub-domain boundary element method.

2 Mathematical Modeling of Moving Interface Flow

As shown in Fig. **1**, Ω_1 and Ω_2 denote the portions of flow domain occupied by fluids 1 and 2, respectively. The fluid regions are separated by a sharp interface. Here, the subscript i denotes each flow region. In addition, ρ denotes the density of fluids. In this model, we assume the existence of velocity potentials $\Phi_i(x,y,t)\ (i=1,2)$ in the fluids both sides of the interface. Then, the governing equations for the velocity vector $\boldsymbol{u_i}=\boldsymbol{u_i}(u_i,v_i)$ are given as follows:

$$\nabla^2\Phi_i=\frac{\partial^2\Phi_i}{\partial x^2}+\frac{\partial^2\Phi_i}{\partial y^2}=0 \quad \text{in} \quad \Omega_i(i=1,2) \tag{1}$$

$$\boldsymbol{u_i}=\nabla\Phi_i \quad \text{in} \quad \Omega_i(i=1,2) \tag{2}$$

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where $\nabla=(\partial_x,\partial_y)$ and ∇^2 denotes the two-dimensional Laplacian. There are two kinds of the boundary conditions to be prescribed. The first one is the wall boundary condition given by:

$$\frac{\partial \Phi_i}{\partial n}=\hat{\boldsymbol{n}}\cdot\nabla\Phi_i=0 \quad \text{on} \quad \Gamma^i_\omega(i=1,2) \tag{3}$$

where $\hat{\boldsymbol{n}}$ denotes the outward unit normal vector on the boundary. The other is the so-called moving boundary conditions on the moving interface Γ_I. They are the kinematic and dynamic conditions. As the mathematical expressions of these conditions, we introduce its Lagrangian description in terms of the Lagrangian coordinates (ξ_i,η_i) for a marked particle on the moving interface. Consequently, the liquid particles on the interface must move with the interface in each domain. Then the kinematic conditions for a particle are given by:

$$\frac{D\xi_i}{Dt}=u_i=\frac{\partial \Phi_i}{\partial x} \quad on \quad \Gamma^i_I(i=1,2) \quad ,$$
$$\frac{D\eta_i}{Dt}=v_i=\frac{\partial \Phi_i}{\partial x} \quad on \quad \Gamma^i_I(i=1,2) \tag{4}$$

where D/Dt is used to express the Lagrangian derivative. Next, we also can express the dynamic condition derived from Bernoulli's equation as the following equation on rate of change of Φ_i :

$$\frac{\partial \Phi_i}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial \Phi_i}{\partial x}\right)^2+\left(\frac{\partial \Phi_i}{\partial y}\right)^2\right\}+g\eta_i+\frac{P_i}{\rho_i}=0 \quad on \quad \Gamma^i_I(i=1,2) \tag{5}$$

where g is the acceleration of gravity, and P_i are the pressure on the interface Γ^i_I . The interfacial conditions should be introduced to this model. To require that two fluids do not separate or cross over at the interface, we must set the following kinematic condition:

$$\frac{\partial \Phi_1}{\partial n}=-\frac{\partial \Phi_2}{\partial n} \quad on \quad \Gamma^i_I(i=1,2) \tag{6}$$

Next, the normal stress of the fluid is to be continuous at the interface. For an inviscid fluid, this means satisfaction of the following dynamical condition that the pressure is continuous:

$$P_1=P_2 \quad on \quad \Gamma^i_I(i=1,2) \tag{7}$$

In this paper, we consider the mathematical model given by equations (1)-(7) as the coupled problem of the boundary-value problem of Laplace equation (1) and the initial-value problem of the system of evolutional equations (4) and (5).

3 Theory of Domain Decomposition Approach for Moving Interface in Flow Region

Let us consider the two layer flow with a moving interface in a domain Ω , which is decomposed into two sub-domains Ω_1 and Ω_2 as shown in Fig. **2**. Here, we can easily transform the field equation (1) into the following boundary integral equation by taking into consideration with the linearity of Laplace equation (1):

$$\int_{\Gamma_i}\Phi_i(\boldsymbol{x})\frac{\partial\omega^*_i}{\partial n}(\boldsymbol{x},\boldsymbol{y})d\Gamma(\boldsymbol{x})=\int_{\Gamma_i}\omega^*_i(\boldsymbol{x},\boldsymbol{y})\frac{\partial\Phi_i}{\partial n}(\boldsymbol{x})d\Gamma(\boldsymbol{x})(i=1,2) \tag{8}$$

in which ω^* is the well-known fundamental solution given by:

$$\omega^*(\boldsymbol{x},\boldsymbol{y})=\frac{1}{2\pi}\ln\frac{1}{r} \quad on \quad r=\|\boldsymbol{x}-\boldsymbol{y}\| \tag{9}$$

If Dirichlet data on the moving interface Γ_I is known, then we can determine its derivative on Γ_I with solution of the above boundary integral equation. In order to solve approximately (8), we use the Boundary Element scheme.

In Domain Decomposition approach of (1), (6) and (7), several formulations can be derived according to treatment of inter boundary conditions of (6) and (7). In this study, we employ the continuity of Dirichlet data (i.e., velocity potential) and Neumann data (i.e., normal velocity) as follows:

$$\Phi_2=\alpha\Phi_1+\beta \quad on \quad \Gamma_I \tag{10}$$

$$\frac{\partial \Phi_1}{\partial n}+\frac{\partial \Phi_2}{\partial n}=0 \quad on \quad \Gamma_I \tag{11}$$

To treat the inter sub-domain boundary condition, the Lagrange multiplier λ is introduced as follows:

$$\Phi_1=\lambda=\frac{1}{\alpha}\Phi_2-\frac{\beta}{\alpha} \quad on \quad \Gamma_I \tag{12}$$

Applying the above conditions to (8), the following inverse formulation is derived:

$$\sum_{i=1}^{2}\left[\int_{\Gamma^i_{I+\omega}}\Phi_i\frac{\partial\omega_i^*}{\partial n}d\Gamma-\int_{\Gamma^i_{I+\omega}}\omega_i^*\frac{\partial\Phi_i}{\partial n}d\Gamma\right]+\int_{\Gamma_I}\left(\frac{\partial\Phi_1}{\partial n}+\frac{\partial\Phi_2}{\partial n}\right)\delta\lambda\, d\Gamma=0 \quad (13)$$

4 Uzawa's Algorithm for Domain Decomposition Approach

Equation (13) consists of the usual boundary integral forms for the sub-domain Ω_i and the constrain term derived from the energy equilibrium as well as the normal velocity continuity among sub-domains. Uzawa's method [4], which is one of iterative solution techniques, is employed here to solve (13). Uzawa's algorithm is summarized as follows:

• *Step 1: Initialization*

$$\lambda^0=\hat{\lambda}(:constant) \quad (14)$$

• *Step 2: Computation in each sub-domain:*

$$\sum_{i=1}^{2}\left[\int_{\Gamma^i_{I+\omega}}\Phi_i\frac{\partial\omega_i^*}{\partial n}d\Gamma-\int_{\Gamma^i_{I+\omega}}\omega_i^*\frac{\partial\Phi_i}{\partial n}d\Gamma\right]+\int_{\Gamma_I}\tau^n\lambda^n d\Gamma=0 \quad (15)$$

where τ denotes the residual value which is the continuity of flux. In this step, we solve the Laplace equations under following boundary conditions:

$$\tau^n=\frac{\partial\Phi_1}{\partial n}+\frac{\partial\Phi_2}{\partial n} \quad on \quad \Gamma_I \quad (16)$$

where superscript *n* indicates the *n*-th iterative step.

$$\frac{\partial\Phi_i}{\partial n}=0 \quad on \quad \Gamma^i_\omega \quad (17)$$

$$\phi_1=\lambda^n, \quad \phi_2=\alpha\lambda^n+\beta \quad on \quad \Gamma^i_I \quad (18)$$

where superscript n indicates the n^{th} iterative step.

•*Step 3: Modification of Lagrange multiplier* λ_n :

$$\lambda^{n+1}=\lambda^n+\omega\tau^n \quad (19)$$

where omega denotes the convergence coefficient.

• *Step 4: Judgement of convergence*

The criterion for convergence employed here is:

$$\int_{\Gamma_I}\frac{\tau^n\cdot\tau^n}{\tau^0\cdot\tau^0}d\Gamma\le\epsilon \quad (20)$$

If λ^n has not converged yet, return to *step 1* by setting $n+1\to n$. By implementation of the above iterative method, we can get the potential and $(\partial\Phi_i/\partial n)^{k+1}$ on the Γ^i_I and use these values for estimation of interfacial dynamics.

5 Formulation for Moving Interface Computations

We will determine the particles on the interface whose velocities $(\partial_t X_I,\partial_t Y_I)$ are a mean values of velocities of the two fluids. The kinematic condition (4) is modified to the following forms as given by:

$$\frac{\partial X_I}{\partial t}=\frac{1}{2}\left[(1+\beta)\frac{\partial\Phi_1}{\partial x}+(1-\beta)\frac{\partial\Phi_2}{\partial x}\right], \quad \frac{\partial Y_I}{\partial t}=\frac{1}{2}\left[(1+\beta)\frac{\partial\Phi_1}{\partial y}+(1-\beta)\frac{\partial\Phi_2}{\partial y}\right] \quad (21)$$

where β is the constant in which $\beta=+1$ corresponds to the lower fluid, $\beta=-1$ is to the upper fluid and $\beta=0$ is to mid-interface particles. In this computation, we adopt $\beta=0$, and the velocity of interface is mid-interface particles of the layer. This system to be considered as the one of first-order ordinary differential equations can be solved approximately by using the time integration scheme. Applying the Euler scheme to the above system, we can determine the new value of ξ and η at the $(k+1)^{th}$ time step. The procedure can be repeated to track the time history of the interface movement.

6 Numerical Experiments and Evaluations

In order to examine applicability of our method proposed, we show the obtained numerical results. We simulate the motion of two different density fluids in which are stratified for the vertical direction under gravitational force g. Two fluids are settled in the rectangular container with non-dimensional width, $L=0.04$ and height, $H=0.06$. And, the container is filled with the lower fluid to a height, $h=0.03$ at the stationary state. This interface is initially flat, but a perturbation is supplied by specifying the y-coordinate component of its position at the interface as $\delta y = A_0 \cos(\pi x / L)$. Numerical computations are carried out for the case given by parameters such that $A_0 = 0.0001$, $g = 1.0$ and $\Delta t = 0.005$. The fluid domains is divided into 50 boundary segments and both interface parts are divided into 20 segments, respectively. In Figure **3**, we show the profiles of interface at each time step in the case of $\rho_1 / \rho_2 = 1.0/2.0$ as density ratio of the two fluids. The perturbation drives the unstable fluid interface, causing it to flow down along the right edge of the box in the form of a fluid spike, while a bubble moves up along the left box edge. Figure **4** shows the profiles of deformed interface at three cases at different time. Figure **5** shows the good convergence of Uzawa's iteration in Case I. Figure **6** shows the situations of convergence using Uzawa's method at each deformation level of interior-interface of domain decomposition computations. From this results, we can recognize the convergence speed of Case I is faster than the speed of Case II or Case III.

7 Conclusion

In conclusions, we have shown the applicability for BE analysis with DDM to numerical simulation for moving boundary problems. We introduced the DDM which is based on construction of the set of BEM for each sub-domain in conjunction with compatibility and equilibrium conditions on the interface. Uzawa's method is effective to iterative computations for this type problem. This solution procedure can simulate the interfacial movement of density stratified flow. Obtained results show the tendency to increase of iteration number in computation at complicated shape of the internal boundary. Consequently, this Domain Decomposition-BE procedure will contribute to the establishment of parallel BEM computation for further applications.

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Figures and Graphs

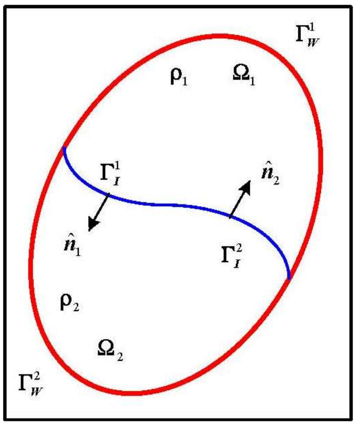

Figure 1: Geometrical configuration of problem.

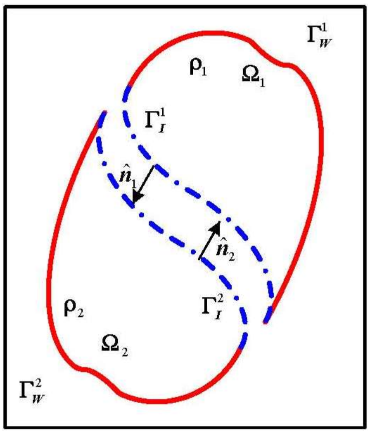

Figure 2: Problem splitted into two sub-domains.

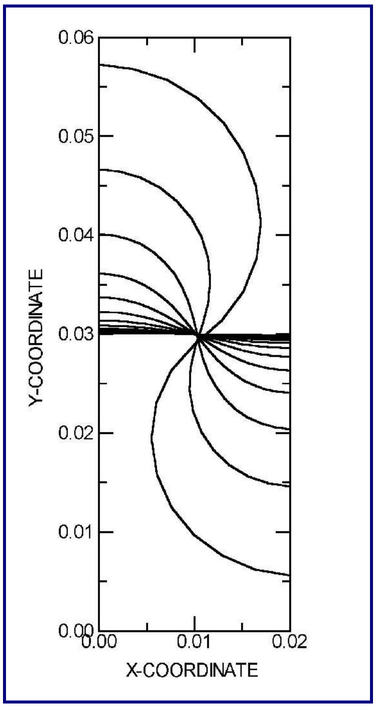

Figure 3: The schematic histories of time-dependent behavior in a moving boundary phenomenon.

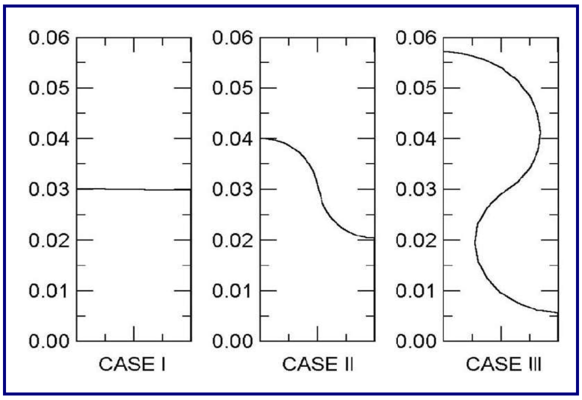

Figure 4: Three profiles showing interface deformations at different out put times. Case I :small deformation at t=0.000sec. Case II :middle deformation at t=0.130sec. Case III :large deformation at t=0.165sec.

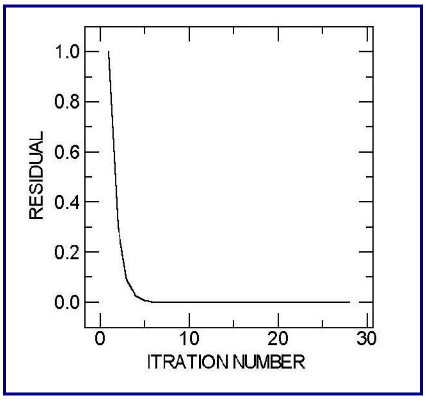

Figure 5: Convergence process of Uzawa's iterations at Case I.

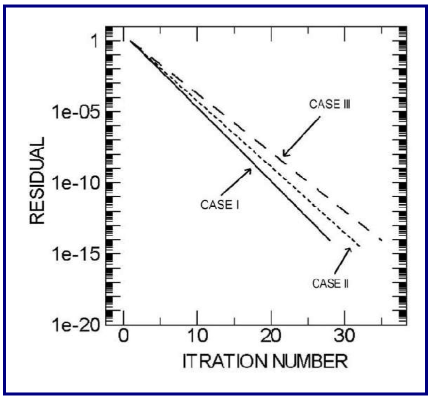

Figure 6: Comparison of convergence situations using DDM applied to three cases.