POD AND MMD COMPARISON IN EXTRACTION OF THE COHERENT STRUCTURES

FEREYDOUN SABETGHADAM
Assistant Professor, Aerospace Engineering Dept.,
Science and Research branch, Azad University, Tehran, Iran
e-mail: fsabet@dpimail.net

Abstract:- In this paper, the coherent structures of a random-nonrandom field are extracted using the Mean Modal Decomposition(MMD) method. As the basis functions, the classical Fourier modes as well as the Proper Orthogonal Decomposition(POD) modes are used as the eigenmodes in the Galerkin projection. At first, the MMD method is introduced and the place of method in the hierarchy of semi-deterministic methods(SDM) and its connection with the other methods especially the POD are described. Then as an example of application of the method, a random-nonrandom field which is obtained from the solution of the Kuramoto-Sivashinsky(KS) equation is analyzed. The KS equation is solved numerically by the spectral Galerkin method to obtain a spatio-temporal chaotic field. The appropriate ensemble is constructed and both the Fourier and the POD eigenmodes are obtained. The ensemble averaging are taken for both eigenmodes and their resultant spectrum are compared. Results show the ability of the method to capturing the mean behavior of the coherent structures and sensitivity of the method to choosing the basis functions.

Key-Words:- Turbulence; Coherent structures; Ensemble averaging; POD; MMD.

1 Introduction

In the second half of the last century, gradually the concept of the coherent structures was appeared in the literature of fluid mechanics. Now after more than a half of a century, finding suitable analysis methods for these unsteady, high energetic, large scale structures is still one of the most challenging issues in the turbulent flows area. In this context, one of the most successful methods has been the decomposition of the field into the coherent and non-coherent parts. The basic idea of such decomposition first arrived in the work of Reynolds and Hussein[1], where they suggested a triple decomposition for each quantity of the field. Their coherent part was obtained from a phase or conditional averaging. Such a decomposition, as has been shown by Ha Minh[2], make a good framework to capturing the mean behavior of the coherent structures. Based on this decomposition, many methods which now, after Ha Minh, are called the Semi Deterministic Methods(SDM) are appeared. various SDMs are arrived in the literature with different names such as the Unsteady Reynolds Averaged Navier Stokes(URANS), Transient Reynolds Averaged Navier Stokes(TRANS), Very Large Eddy Simulation(VLES), and so on. These methods more or less are based on an ensemble averaging process. Although the ensemble averaged Navier Stokes equations are the same as the RANS equations, however the types of modelling have to be changed[3, 4, 5]

On the other hand, another well known method for capturing the coherent motions is the Proper Orthogonal Decomposition(POD) method, first introduced by Lumley[6]. In the POD method, the eigenfunctions of some first larger eigenvalues of the autocorrelation tensor are found and in this manner, the most energetic modes of the field are captured. Then the whole of the ensemble can be reconstructed approximately by a Galerkin projection which its eigenmodes are the POD modes [7].

Recently the Mean Modal Decomposition (MMD) method has been suggested as a new method for solution of the turbulent flows by the author[8]. Essentially the method consists of projection of the field, collected in an appropriate ensemble, into an appropriate function space and then taking an ensemble averaging on each mode in that space. The number of modes, which somehow determines the desired level of deterministic computations, is preassumed and in this manner, the number of coupled equations that should be solved will be determined. In this method all the non-random modes, in the sense of the ensemble averaging, are resolved and all the random modes should be modeled. Therefore the method could be placed into the semi-deterministic methods. One of the most advantages of this method in comparison with some of the other similar methods, is that there is a straight forward way to derivation of the mean modal equations from the governing equations of the phenomenon[8]. On the other hand, it seems that the ensemble averaging process do not vanish the coherent and semi-coherent motions [2, 5]. So it is expected that this method could capture the coherent structures from a random-nonrandom field.

The projection could be done in spatial or temporal fashions. The Spatial Mean Modal Decomposition (SMMD) results in an evolution system of equations whereas the Temporal Mean Modal Decomposition (TMMD) yields to a set of equations which produces a periodic approximation of the field[8].

In this paper, the MMD method and its main properties are presented firstly. Then the spectral Galerkin method is used to direct numerical solution of the Kuramoto-Sivashinsky equation to obtain a spatio-temporal chaotic field with dominant coherent structures. Using these data, an appropriate ensemble is generated. This ensemble, then is analyzed with the MMD method. For the POD analysis, the Singular Value Decomposition(SVD) in conjunction with the method of snapshots is used and for the MMD eigenmodes, both the POD modes and the classical Fourier modes are used. Finally the field and the spectrum of the approximate fields are compared with each other.

2 Mean Modal Decomposition and Averaging

To emphasize the methodology and avoid complexities of multi dimensional formulations, the formulation of the mean modal decomposition for one dimensional scalar functions is presented in this section. As will be shown, almost all the major characteristics of the method are observable in the one dimensional formulation. However, the formulations for spatio-temporal multi dimensional fields and its various forms will present in subsequent sections. Consider an ensemble $\{u^m\}_{m=1}^M$ of a scalar field $u^m = u^m(t)$ with M elements, describing a more or less non periodic oscillating repeatable phenomenon and $t \in [t_0, t_0 + T]$ is the independent variable, say time. On the $[t_0, t_0+T]$ interval the inner product is defined as

$$(u, v) = \frac{1}{T}\int_{t_0}^{t_0+T} uv^\dagger dt \tag{1}$$

in which $()^\dagger$ denotes the complex conjugate. Also the ensemble averaging is defined as

$$\langle u\rangle(t) = \frac{1}{M}\sum_{m=1}^{M} u^m(t) \tag{2}$$

We want to provide a unique representation for the whole of the ensemble which meanly show the behavior of the nonrandom,i.e. the coherent part of the field. In this context, the mean modal representation of the ensemble $\{u^m(t)\}_{m=1}^M$ with the mean fundamental frequency $\omega = \frac{2\pi}{T}$ until N terms is defined as

$$\langle u\rangle_{\omega,N}(t) = \sum_{n=-N}^{+N} \bar{u}_n \varphi_n(t) \tag{3}$$

where $\varphi_n(t)$ s are orthonormal functions on $[t_0, t_0 + T]$ and the coefficients $\bar{u}_n$ are obtained from the application of the mean modal operator $\bar{\mathcal{F}}_n\{\}$ on ensemble $\{u^m\}$ as

$$\bar{u}_n = \bar{\mathcal{F}}_n\{u^m\} = \langle (u, \varphi_n)\rangle \tag{4}$$

Note that the mean modal operator has an ensemble averaging process for each mode and therefore it is expected to discard the random and non periodic motions for each mode. Then each element of the ensemble $\{u^m(t)\}_{m=1}^M$ can be reconstructed by definition of a two part decomposition as

$$u^m(t) = \langle u\rangle_{\omega,N}(t) + u'^m_{\omega,N}(t) \tag{5}$$

It can easily be shown that according to above definitions, the fluctuation term $u'^m_{\omega,N}$ has the property

$$\bar{\mathcal{F}}_n\{u'^m_{\omega,N}\} = 0 \tag{6}$$

It is a well-known classical property of the modal decomposition that for the stationary random processes, the mean amplitude of the process is zero (Pope, 2002). Therefore the decomposition (5) apart each involving mode into random and nonrandom and assign the mean of the nonrandom part into the mean modal term. It means that the fluctuation part in (5), contains the background random motion, and has the property that its ensemble average for each mode are vanished. As will be shown, this fundamental property make decomposition (5) such that it can directly imposed in the governing differential equations, say the N-S equations and it is one of the advantages of this method to some other methods. In the next section the main properties of the mean modal decomposition is presented.

2.1 Properties of the Mean Modal Decomposition

Without dealing the details of derivations, the main important properties of the decomposition and the mean modal operator that will be needed to implement ion of the decomposition into the N-S equations are listed and explained shortly here. According to our final goal, i.e. implementation of the decomposition into the N-S equations, our focus will be on that properties of the mean modal operator (4) which will be needed in the derivation. Obviously this list is not complete and a detailed investigation should be contained its other important mathematical properties.

1. The mean modal operator is linear. If we have two different scalar fields u and v in each ensemble element, say different velocity components, then we have

$$\bar{\mathcal{F}}_n\{\alpha u^{\mathrm{m}} + \beta v^{\mathrm{m}}\} = \alpha\bar{\mathcal{F}}_n\{u^{\mathrm{m}}\}+\beta\bar{\mathcal{F}}_n\{v^{\mathrm{m}}\} = \alpha\bar{u}_n+\beta\bar{v}_n \tag{7}$$

   in which α and β are not functions of the independent variable.

2. To determination of the mean modal representation of the derivative with respect to the independent variable, specification of the function space will be needed. If for instance assuming a Fourier space, then

$$\bar{\mathcal{F}}_n\{\frac{\partial u^{\mathrm{m}}}{\partial t}\} = in\bar{\mathcal{F}}_n\{u^{\mathrm{m}}\} = in\bar{u}_n \tag{8}$$

3. The mean modal operator can be applied on the mean modal approximation and the result is the same as its operation on the whole of the ensemble

$$\bar{\mathcal{F}}_n\{\langle u\rangle_{\omega,N}\} = \bar{\mathcal{F}}_n\{u^{\mathrm{m}}\} = \bar{u}_n \tag{9}$$

4. The mean modal of the product of two mean modal approximations in each mode, represents the effects of all other modes on the mode which is considered

$$\bar{\mathcal{F}}_n\{\langle u\rangle_{\omega,N}\langle v\rangle_{\omega,N}\} = \underbrace{\sum_{\mathrm{j}=-N}^{+N}\sum_{\mathrm{l}=-N}^{+N}}_{\mathrm{j+l}=n}\bar{u}_{\mathrm{j}}\bar{v}_{\mathrm{l}} = \sum_{\mathrm{j}=-N}^{N}\bar{u}_{\mathrm{j}}\bar{v}_{n-j} \tag{10}$$

5. In spite of the Reynolds averaging, the mean modal of the mixed fluctuating and mean modal approximation will not be zero and as will be mentioned, should be modeled in the resulting equations

$$\bar{\mathcal{F}}_n\{v'^{\mathrm{m}}_{\omega,N}\langle u\rangle_{\omega,N}\} \neq 0 \tag{11}$$

All above properties but (8) are valid for any orthonormal functions defined on the domain $t \in [t_0, t_0+T]$. The mean modal decomposition can be directly applied into the N-S equations using these properties. As will be shown, in spite of the RANS equations, the results are a system of coupled, nonlinear, non closed of PDEs that have some terms arising from the nonlinear interaction of the fluctuations and the mean modal term, which should be modeled.

2.2 Connection With the POD Method

The mean modal decomposition as defined here, is a two part decomposition. Therefore each element of the ensemble $\{u\}_{m=1}^{M}$ is decomposed into two parts as

$$u^m(t) = u_a(t) + u'^m(t) \tag{12}$$

in which u_a, i.e. the mean modal representation, is the approximation part which is expected to be a representation for whole of the ensemble $\{u^m\}_{m=1}^{m}$ and u'^m, i.e. the fluctuation part, is the deviation of the approximation part from the exact values in each ensemble element. Now projection of the above relation on the nth mode, i.e. inner product of the relation with $\varphi_n(t)$, yields to

$$u_n^m = (u_a)_n + (u'^m)_n \tag{13}$$

This is the modal form of relation (5) and easily could be shown that is a sufficient condition for it and is valid for all the modes involved in the field. The right hand side of relation (13) have two variables. Therefore for each mode there is one equation for two variables. To unique determination of the variables one additional relation or restriction is needed. For example, in the Reynolds decomposition, this additional relation is provided by the

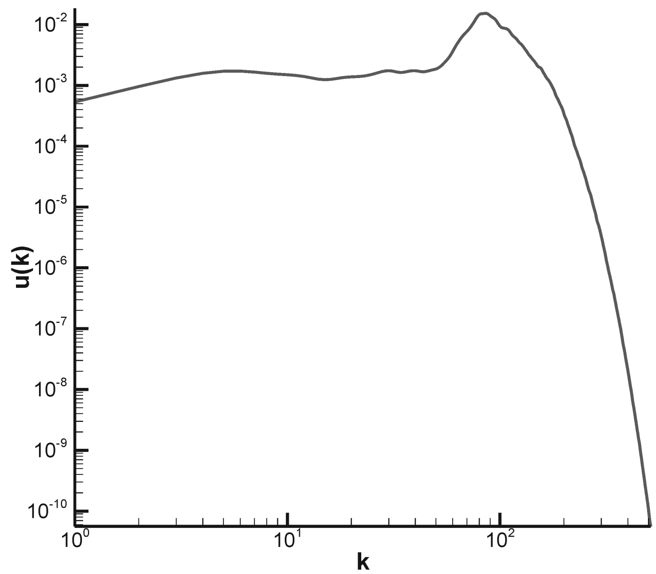

Figure 1: The power spectrum (root mean square) of u. The spectrum is obtained from time history. Because of quadrature nature of the r.m.s, time history can be replaced the ensemble averaging without any serious error.

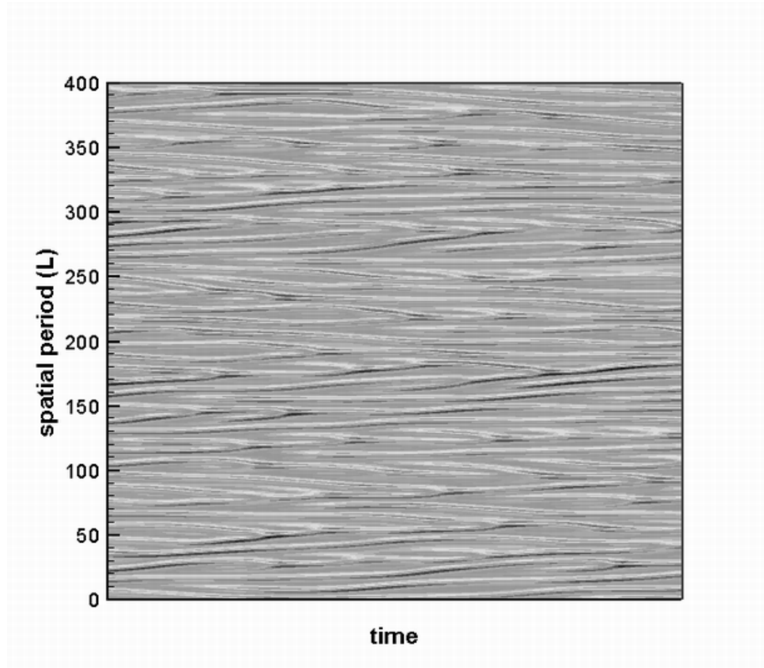

Figure 2: Spatio-temporal representation of the u for about 100 seconds. The boundary conditions are periodic and $L = 400$. The presence of the coherent structures is obvious.

fact that the mean of the fluctuation part is vanished. In mean modal decomposition the restriction is constructed according to the relation (6) which means that the ensemble average of $(u'^m)_n$ is vanished for all modes, i.e.

$$\langle (u'^m)_n \rangle = \frac{1}{M} \sum_{m=1}^{M} (u'^m)_n = 0 \tag{14}$$

which results in

$$(u_a)_n = \frac{1}{M} \sum_{m=1}^{M} (u^m)_n \tag{15}$$

which is the relation (4).
On the other hand, the POD method can be viewed as a two part decomposition just like the relation (12) with another restriction. In fact in the POD method, the restriction is directly applied to the approximation part such that

$$\max \frac{\langle |(u_a, \varphi_n)|^2 \rangle}{\|\varphi_n\|^2} \tag{16}$$

with our notation and with respect of our assumption about orthonormality of the functions φ_n, the above relation can be written as

$$\max\{\frac{1}{M} \sum_{m=1}^{M} [(u^m)_n - (u'^m)_n]^2\} = \max\{\frac{1}{M} \sum_{m=1}^{M} (u_a^m)_n^2\} \tag{17}$$

In such interpretation of the POD method, relations (13) and (17) determines the approximate representation of the field uniquely. In fact in the POD method in the notation of relation (13), we want to find the approximation part such that the ensemble average of the square of $(u_a)_n$ be maximized.
As one can see, the mean modal approximation and the POD approximation, both get an orthogonal expansion representation of the field but the coefficients of the POD representation are obtained such that the most energy of the field be captured This is because of the power of the norm in the relation (16) which is a measure of the kinetic energy.

2.3 Extraction of the Coherent Structures

As a first real application of the MMD method to extraction of the coherent structures, to avoid the complexities of multidimensional problems and the boundaries, which usually forced to do a kind of modelling, the one dimensional KS equation with periodic boundary conditions has been chosen to produces a spatio-temporal field with dominant coherent structures. The one dimensional

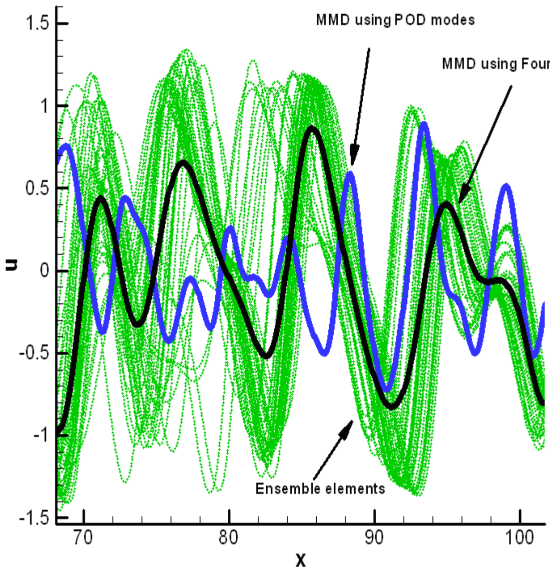

Figure 3: Comparison of Fourier and POD mean modal decomposition in extraction of the coherent structures. Note on the phase of the Fourier modes with the original field.

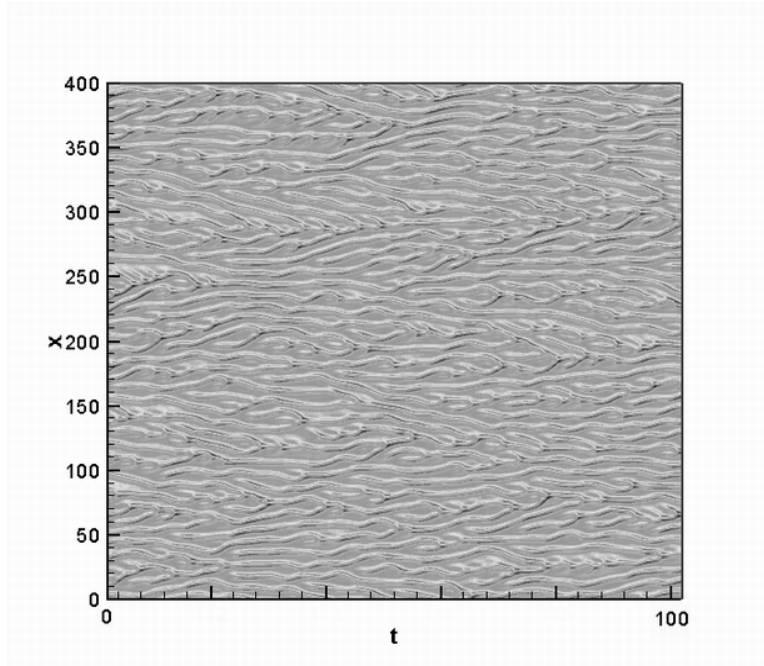

Figure 4: Field obtained from the mean modal decomposition method with the POD eigenmodes. note to the wave number(wave length) of the coherent structures in comparison to their time duration.

drift free KS equation is defined as[7]

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0 \qquad L \leq x \leq 0 \tag{18}$$

with periodic boundary conditions

$$u(0) = u(L) \;\; ; \;\; u_x(0) = u_x(L) \;\; , \;\; \ldots \tag{19}$$

In such definition of the KS equation, the spatial period length scale 'L' plays the rule of the bifurcation control parameter and for enough large 'L', we are faced with spatio-temporal chaotic field very similar to high Reynolds number NS turbulence[7].

Although $L > 10\pi$ produces more or less appropriate results, however, to following the work of Berkooz *etal*[10] and to checking the results, and also to produce a field as similar as possible to the NS turbulent fields, in the present work, the KS equation is solved for $L \approx 400$ by the spectral Galerkin method with 512 eigenmodes. The resulting space-time field and the power spectrum of the solution are presented in figures(1) and (2).

Then an ensemble with 20 elements, each includes 100 seconds time history were produced from a nominally the same initial conditions. To produce each element, the 300 first Fourier modes were forced to be nonrandom, i.e. read from the same file, in each run and the remainder of the modes were read from the last time step from the last run which somehow get a random nature to the high frequency modes but with more or less the same spectrum as a run with all the modes nonrandom. It is important to note that because of the difference between the time scales of the repeated and random modes, we needed a settling time until the small scales can affect the large scales. For the present case, the 300 Fourier modes repeated, this settling time was about 2.5 seconds which in comparison of all the 100 seconds is almost negligible.

For the mean modal analysis, each element of the ensemble then projected into a number of modes (from 10 to 150 in various runs) and then the average of these modes were calculated to obtain a mean modal representation for the whole of the ensemble.

On the other hand, for the POD calculations, in each time step, the 20 element ensemble is used to construct the autocorrelation matrix. Then the method of snapshots is used to reduce the dimension of the system and the SVD method is used to find the eigenvalues and eigenvectors. Then by the use of these eigenfunctions, the approximate representation of each ensemble element is done and the power spectrum of this approximate representation is calculated. The figures (3) and (4) show some of the results.

In the Fig.(3) a part of the space domain in $t = 80$

is shown as an example. The following items could be observed

a) Both the Fourier and the POD mean modes generally are in agreements in the capturing the dominant part of the motion with more or less the same mean amplitudes.

b) In the reconstructed field with the Fourier modes, obviously a better phase capturing is achieved. It is so in the other parts of the field.

c) The ensemble averaging process, is decreased the mean amplitudes of the POD eigenmodes. In the opinion of the author, it is done because of the quadratic nature of the optimization process in the POD(we maximize the square of the amplitudes) which avoid discarding of the random parts.

Figure(4) shows the field obtained from the Fourier mean modal method. First of all, the presence of the coherent structures is shown again. However in comparison with the Fig.(2), although the wave number of the coherent structures is captured correctly, as we saw in the power spectrum also, however their frequency, i.e. their time durations, are different. In fact they are seems truncated in time. In opinion of the author it is because of the method of construction the ensemble. In fact some numerical experiments with different percent of random and nonrandom modes shows obvious effect on the time scale of the coherent structures.

3 Conclusion

The KS equation is solved numerically to construct an ensemble which is used to analysis of the ability of the MMD method to extraction of the coherent structures. For mean modal analysis both classical Fourier modes and the POD eigenmodes are used. The results shows good ability of the method to capturing of the coherent structures and sensitivity of the method to the chosen basis functions.

References

[1] Hussein, A. K. M. F., Reynolds, W. C. The Mechanics of an Organized Wave in Turbulent Shear Flow. *J. Fluid Mech.*, 1972, 54 (2), 241-261

[2] Ha Minh, H., Kourta, A. Semi-deterministic Turbulence Modelling for Flows Dominated by Strong Organized Structures. In: *Proceedings 9th symposium on turbulent shear flows*, 1993, Kyoto, Japan.

[3] Aubrun, S., Kao, P. L., Bossin, H. C., Experimental Coherent Structures Extraction and Numerical Semi-Deterministic Modellingn in the Turbulent Flow Behind a Backward-facing Step, *Experimental Thermal and Fluid Science*, 2000, 22, 93-101.

[4] Osterland, J. M., Lindgren, B., Johanson, V. A., Flow Structures in Zero Pressure-gradient Turbulent Boundary Layers at High Reynolds Numbers. European Journal of Mechanics/B Fluids, 2003, 22, 379-390.

[5] Bonnet, J. and Delville, J., Review of Coherent Structures in Turbulent Free Shear Flows and their Possible Influence on Computational Methods, *Flow, Turbulence and Combustion*, 2001, **66**: 333-353

[6] Lumley, J. L. the structure of inhomogeneous turbulence. In A. M. Monin and V. I. Tartarski(Eds.) Atmospheric turbulence and wave propagation, 1967, pp. 166-178. Moscow: Nauka.

[7] Holmes,P.,J. L. Lumley and G. Berkooz. *Turbulence, Coherent Structures, Dynamical systems and Symmetry.* 1996, New York: Cambridge University Press.

[8] Sabetghadam, F. Generalization of the RANS equations Using Mean Modal Decomposition of the Navier-Stokes Equations. ESDA2004-58395, proceedings of ESDA 2004, Manchester, England.

[9] Pope, S. B. *Turbulent Flows.* 2002, Cambridge University, Press.

[10] Berkooz, G., Elezgaray, J., Holmes, P., Coherent Structures in Random Media and Wavelets, Physica D, 1992, 47-58.