On the Approximation Properties of $q$-Laguerre type Modification of Meyer König and Zeller Operators

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Abstract: - In the present paper, we introduce a Laguerre type positive linear operators based on the $q$-integers including the $q$-Meyer König and Zeller operators defined by Doğru and Duman in [7]. Then we obtain some results about Korovkin type approximation properties and rates of convergence for this generalization.

Key-Words: - Positive linear operators, $q$-Meyer König and Zeller operators, $q$-Laguerre polynomials, modulus of continuity.

1 Introduction
The following operators were introduced by Meyer-König and Zeller [11]:

$$M_x^*(f;x) = \sum_{k=0}^{\infty} f\left( \frac{k}{k+n+1} \right) m_{n,k}(x) \quad (0 \leq x < 1) \quad (1.1)$$

where

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^n.1$$

To obtain the monotonicity properties of the operators (1.1), Cheney and Sharma [4] were introduced the following operators:

$$M^n_x(f;x) = \sum_{k=0}^{\infty} f\left( \frac{k}{k+n} \right) m_{n,k}(x) \quad (0 \leq x < 1). \quad (1.2)$$

The operators (1.2) are also called as Bernstein power series in the literature.
A generalization of the Meyer-König and Zeller operators has been given by Doğru in [5]. Then a Stancu type generalization of the operators in [5] is defined by Agratini in [1].

Recently, in [2], Altin, Doğru and Taşdelen studied on some approximation properties of a generalization of Meyer-König and Zeller operators by generating functions.
The $q$-type generalization in approximation theory were introduced by Phillips [14] for the classical Bernstein polynomials in 1996. This generalization is obtained by replacing the general binomial expansion with $q$-binomial expansion. The rate of convergence and Voronovskaja type asymptotic estimate are obtained by Phillips and colleagues for this $q$-generalization of Bernstein polynomials. The different convergence properties of this generalization has been obtained by Goodman, Oruç and Phillips [8].
In this point, recalling some definitions about $q$-integers will be suitable:
For any fixed real number $q > 0$, we denote $q$-integers by $[r]$ where

$$[r] = \begin{cases} (1-q^r)/(1-q) & \text{if } q \neq 1 \\ r & \text{if } q = 1. \end{cases} \quad (1.3)$$

Also, $q$-binomial coefficients are defined by

$$\binom{n}{r} = \frac{[n]!}{[r][n-r]!} \quad r = 0,1,...,n,$$

where

$$[r]! = \begin{cases} [r][r-1]...[1] & \text{if } r = 1,2,... \\ 1 & \text{if } r = 0 \end{cases}$$

and $n,r \in N_0$.

It is clear that when $q = 1$, the $q$-binomial coefficients reduce to ordinary binomial coefficients.
In [15], Triff defined the Meyer-König and Zeller operators based on the $q$-integers as follows:

$$M_{n,q}(f;x) = u_{n,q}(x) \sum_{k=0}^{\infty} f\left( \frac{k}{k+n} \right) \binom{n+k}{k} x^k \quad (1.4)$$
for $0 \leq x < a < 1$ where

$$ u_{n,q}(x) = \prod_{k=0}^{n}(1-xq^k). $$

But, unfortunately, it is not possible to obtain the explicit formulae for the second moment of $M_{n,q}(f;x)$. Therefore, in [7] following generalization of the q-Meyer König and Zeller operators is introduced by Doğru and Duman:

$$ M_n(f;q;x) = u_{n,q}(x) \sum_{k=0}^{\infty} f\left(\frac{q^n[k]}{k}\right) \left[\frac{n+k}{k}\right]^k x^k $$

for $0 \leq x < a < 1$.

The $A$-statistical approximation properties of $M_n(f;q;x)$ are investigated in [7]. Moreover, in [7], the rates of $A$-statistical approximation of $M_n(f;q;x)$ to $f(x)$ are estimated by using the modulus of continuity, Peetre $K$-functionals and Lipschitz type maximal functions.

In this study, the $q$-Laguerre type positive linear operators including the operators $M_n(f;q;x)$ are defined and their Korovkin type approximation properties and rates of convergence are investigated.

### 2 Construction of $q$-Laguerre Type Operators

In [4], Cheney and Sharma also introduced the following operators:

$$ P_n(f;x,t) = (1-x)^{a+1} \exp\left(\frac{nt}{1-t}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t)x^k $$

for $0 \leq x < 1$, $-\infty < t \leq 0$ where $L_k^{(n)}(t)$ denotes the Laguerre polynomials and investigated the approximation properties of these operators.

Because of $L_k^{(n)}(0) = \left(\begin{array}{c} n+k \\ k \end{array}\right)$, $M_n(f;x)$ is the special case of the operators $P_n(f;x)$.

In this part, we will define a modification of the operators $P_n(f;x)$ based on the $q$-integers.

The $q$-Laguerre polynomials have the explicit expression (see [9, p.29], [10, p.57] and [12, p.21])

$$ L_n^{(\alpha)}(t;q) = \left(\frac{q^{\alpha+1};q_n}{(q;q)_n}\right) \sum_{k=0}^{n} \frac{(q^{-n};q)_k (k)_2 (1-q)k (q^{\alpha+1};q)_k (q^n;q)_k}{(q^{\alpha+1};q)_k (q;q)_k} $$

where

$$ (x;q)_n = \begin{cases} 1 & \text{if } n = 0 \\ (1-x)(1-xq)...(1-xq^{n-1}) & \text{if } n \in \mathbb{N}. \end{cases} $$

Moak [12] gave the following recurrence relation and generating function for the $q$-Laguerre polynomials

$$ u_k^{(\alpha)}(t;q) = [k+\alpha]q^{-\alpha-k} L_k^{(\alpha)}(t;q) - [k]q^{-\alpha-k} L_k^{(\alpha)}(t;q) $$

(2.2)

$$ F_n(x,t) = \frac{x^{\alpha+1};q_{(x)}^n}{(x;q)_n} \sum_{m=0}^{\infty} \frac{m^{2+\alpha} \bar{m}^{-(1-q)x} m}{(q;q)_m (xq^{-1};q)_m} $$

(2.3)

where $\Re \alpha > -1$, $k = 1,2,...$ and

$$ (x;q)_\infty = \prod_{s=0}^{\infty} (1-xq^s). $$

We consider the sequence of linear positive operators

$$ V_n(f;\cdot;\cdot,q;x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t;q)x^k $$

(2.4)

where $x \in [0,1]$, $t \in (-\infty,0]$, $q \in (0,1]$ and $\{F_n(x,t)\}_{n \in N}$ is the generating functions for the $q$-Laguerre polynomials which was given in (2.3).

If we replace $f\left(\frac{k}{k+n}\right)$ by $f\left(\frac{q^n[k]}{k+n}\right)$ in (2.4), then these operators turns to q-Laguerre type generalization of Trif’s operators which was investigated by Özarslan in [13].

Notice that, since

$$ L_k^{(n)}(0;q) = \left(\frac{mq^n}{q(q^n;q)_k}\right) \left[\frac{n+k}{k}\right], $$

and

$$ \frac{1}{F_n(x,0)} = (x;q)_\infty \sum_{s=0}^{n} (1-xq^s) = u_n,q(x) $$

then the operators in (1.5) is the special case of $V_n(f;\cdot;\cdot,q;x,t)$ for $t = 0$. Also note that, we have

$$ V_n(f;f;\cdot;\cdot,q;x,t) = P_n(f;\cdot;\cdot). $$

To obtain the approximation properties of the operators $V_n(f;\cdot;\cdot,q;x,t)$, we need the following lemmas.
Lemma 2.1. We have

\[ F_{n+1}(x,t) \leq \frac{F_n(x,t)}{(1-aq^{n+1})}. \]

Proof. Since \( q \leq 1 \) and \( (xq^{n+m+1};q)_m \geq (1-aq^{n+1}) \) we get

\[ F_{n+1}(x,t) = \frac{(xq^{n+1};q)_\infty}{(x;q)_\infty} \sum_{m=0}^\infty q^{m+n+m}[-(1-q)x]^m \]

\[ \leq \frac{(xq^{n+1};q)_\infty}{(1-aq^{n+1})(x;q)_\infty} \sum_{m=0}^\infty q^{m+n+m}[-(1-q)x]^m \]

\[ = \frac{F_n(x,t)}{(1-aq^{n+1})}. \]

Lemma 2.2. For all \( n \in N, x \in [0,a](0 < a < 1) \), we have

\[ V_n(s;q;x,t) - x \leq \frac{|x|}{n(1-xq^{n+1})} + (q^n - 1)x. \]

Proof. Using (2.2) in (2.4), we have

\[ V_n(s;q;x,t) = \frac{q^n}{F_n(x,t)} \sum_{k=1}^n \frac{[k]}{[n+k]} L_k^{(n)}(t;q)x^k \]

\[ = \frac{q^n x}{F_n(x,t)} \sum_{k=1}^n \left( \frac{[k]}{[n+k]} t^{(n)}(t;q)x^k \right) x^{k-1}. \]

Since

\[ -x \sum_{k=0}^{n} \frac{q^{k+n+1}}{F_n(x,t)} L_k^{(n+1)}(t;q)x^k \geq 0 \]

then

\[ V_n(s;q;x,t) - x \geq (q^n - 1)x. \] (2.6)

Taking into consideration \([n] \leq [n+k](n,k \in N_0)\) and \(0 < q \leq 1\) in (2.5) and using Lemma 2.1, we get

\[ V_n(s;q;x,t) \leq x - \frac{x}{|n|} F_n(x,t) \]

\[ \leq x - \frac{x}{|n|} \frac{[k]}{[n]}. \] (2.7)

From (2.6) and (2.7) the proof is completed.

Lemma 2.3. For all \( n \in N, x \in [0,a](0 < a < 1) \), we have

\[ V_n(s^2;q;x,t) - x^2 \leq \frac{|x|}{n(1-xq^{n+1})} + \frac{x}{n(1-xq^{n+1})}. \] (2.8)

Proof. From the definition of the operator one can write

\[ V_n(s^2;q;x,t) = \frac{q^{2n}}{F_n(x,t)} \sum_{k=1}^n \frac{[k]}{[n+k]} L_k^{(n)}(t;q)x^k. \] (2.9)

Using the recurrence formula (2.2) twice and the fact that

\[ [k] = [k-1] + q^{k-1}, \]

we can prove that

\[ \left( \frac{[k]}{[n+k]} \right)^2 L_k^{(n)}(t;q) \leq \frac{[k-1]}{[n+k]} L_{k-1}^{(n)}(t;q) \]

\[ - \frac{q^{n+k-1}}{[n+k]} L_{k-1}^{(n)}(t;q) \]

\[ + \frac{q^{k-1}}{[n+k]} L_{k-1}^{(n)}(t;q) \]

\[ - \frac{q^{n+k}[k]}{([n+k])^2} L_{k-1}^{(n)}(t;q). \]

So,

\[ V_n(s^2;q;x,t) - x^2 \leq \frac{q^{2n}}{F_n(x,t)} \sum_{k=2}^{n} \frac{[k]}{[n+k]} L_k^{(n)}(t;q)x^k - x^2 \]

\[ + \frac{q^{2n}t}{F_n(x,t)} \sum_{k=0}^{n} \frac{q^{n+k-1}}{[n+k]} L_k^{(n)}(t;q)x^k \]

\[ + \frac{q^{2n}}{F_n(x,t)} \sum_{k=0}^{n} \frac{q^{k-1}}{[n+k]} L_k^{(n)}(t;q)x^k \]

\[ + \frac{q^{2n}t}{F_n(x,t)} \sum_{k=0}^{n} \frac{q^{n+k}[k]}{([n+k])^2} L_{k-1}^{(n)}(t;q)x^k \]. (2.10)

Thus the right member of (2.10) splits naturally into four parts, which we analysis separately below. Since \(0 < q < 1\) and \([k],[n] \leq [k+n]\), it is obvious that

\[ \frac{q^{n+k}[k]}{([n+k])^2} \leq \frac{1}{[n]} \]

We get, using Lemma 2.1,\n
\[ \frac{q^{2n}}{F_n(x,t)} \sum_{k=2}^{n} \frac{[k]}{[n+k]} L_k^{(n)}(t;q)x^k \leq \frac{|x|}{n(1-xq^{n+1})} \] (2.11)

and

\[ \frac{q^{2n}}{F_n(x,t)} \sum_{k=0}^{n} \frac{q^{n+k-1}}{[n+k]} L_k^{(n)}(t;q)x^k \leq \frac{x^2}{n(1-xq^{n+1})} \] (2.12)

In a similar manner,

\[ \frac{q^{2n}}{F_n(x,t)} \sum_{k=0}^{n} \frac{q^{k-1}}{[n+k]} L_k^{(n)}(t;q)x^k \leq \frac{x^2}{n(1-xq^{n+1})} \] (2.13)

Finally, since \([k-1] \leq [n+k-1]\), we can write

\[ \frac{q^{2n}}{F_n(x,t)} \sum_{k=2}^{n} \frac{[k-1]}{[n+k-1]} L_k^{(n-2)}(t;q)x^k - x^2 \leq 0. \] (2.14)
On the other hand, using the expression
\[ s^2 - x^2 = (s - x)^2 + 2sx - 2x^2, \]
we may write
\[ V_n(s^2; q; x,t) - x^2 = V_n((s - x)^2; q; x,t) + 2xV_n((s - x); q; x,t). \]
Thus from (2.11), (2.12), (2.13) and (2.14) we have (2.8) immediately.

In the proof of these lemmas, we used the similar technique given by Doğru in [5] (see also [6]).

3 Rate of Convergence

In this section, we will compute the rate of convergence of \( V_n(f; q; x,t) \) to \( f(x) \) by means of the classical modulus of continuity.

Let \( f \in C[a,b] \). The modulus of continuity of \( f \) denoted by \( \omega(f; \delta) \), is defined as
\[ \omega(f; \delta) = \sup_{|s-x| \leq \delta, x,s \in [a,b]} |f(s) - f(x)|. \]

It is also well known that for any \( \delta \geq 0 \)
\[ |f(s) - f(x)| \leq \omega(f, \delta) \left( \frac{|s-x|}{\delta} + 1 \right). \]
Notice that, we will use the notation \( \|f\|_{C[0,a]} \) instead of \( \|f\| \) for abbreviation.

Details for modulus of continuities and smoothness can be found in [3].

Theorem 3.1. For all \( f \in C[0,a] \), we have
\[ \|V_n(f; q; x,t) - f\| \leq 2\omega(f, \delta_n) \]
where
\[ \delta_n = \left[ \frac{\|2a + a^2\|}{\left[ n \right]|1 - a^{q+1}|} + \frac{a}{\left[ n \right]} + \left( q^n - 1 \right)a \right]^{1/2}. \]
Proof. Let \( f \in C[0,a] \). By linearity and monotonicity of \( V_n(f; q; x,t) \) to according to \( f(x) \) and using (3.1), we obtain
\[ V_n(f; q; x,t) - f(x) \leq V_n\left[ \frac{|f(s) - f(x)|}{q; x,t} \right] \]
\[ \leq \omega(f, \delta_n) V_n\left[ 1 + \frac{|s-x|}{\delta_n}; q; x,t \right] \]
\[ = \omega(f, \delta_n) \left[ 1 + \frac{1}{\delta_n} \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{q^n[k]}{[k+n]} - x^t \left( f^{(n)}(x; q) x^t \right)^{1/2}. \]
By the Cauchy - Schwarz inequality we have
\[ V_n(f; q; x,t) - f(x) \leq \omega(f, \delta_n) \left[ 1 + \frac{1}{\delta_n} \left( V_n\left( \frac{(s-x)^2; q; x,t} \right) \right)^{1/2}. \]
This implies that
\[ \|V_n(f; q; x,t) - f\| \leq \omega(f, \delta_n) \left[ 1 + \frac{1}{\delta_n} \left( V_n\left( \frac{(s-x)^2; q; x,t} \right) \right)^{1/2}. \right. \]
\[ \times \sup_{x \in [0,a]} \left( V_n\left( \frac{(s-x)^2; q; x,t} \right) \right)^{1/2}. \]

For each \( x \in [0,a] \), one can write
\[ V_n((s-x)^2; q; x,t) \leq \left( V_n\left( \frac{(s-x)^2; q; x,t} \right) - x^2 \right) \]
\[ + 2xV_n\left( \frac{(s-x)^2; q; x,t} \right) - x^2 \]
So, by Lemma 2.2 and Lemma 2.3 we get
\[ \sup_{x \in [0,a]} \left( V_n\left( \frac{(s-x)^2; q; x,t} \right) \right)^{1/2}. \]
\[ \leq \frac{a}{\left[ n \right]|1 - a^{q+1}|} + \frac{a}{\left[ n \right]} + \left( q^n - 1 \right)a \]
and combining (3.2) with (3.3), the proof is completed.

Remark 3.2. Since \( \delta_n \to 0 \) as \( n \to \infty \), under the assumption \( t \to 0 \), we obtain a rate of convergence for \( V_n(f; q; x,t) \) by Theorem 3.1.

References: