Transpof the Weighted Mean Matrix on Weighted Sequence Spaces

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Abstract: In this paper, we concern with transpose of the weighted mean matrix (This is upper triangular matrix.) on weighted sequence spaces \( \ell_p(w) \) and \( d_p(w) \) which is considered by the author in [8] and [9] for special case of these operator, such as Copson on \( \ell_1(w) \) and \( d(w,1) \). Also, in a recent paper [7], the author has discovered the upper bound for the Copson operator on the weighted sequence spaces \( d(w, p) \). Also, we establish analogous upper bound for the continuous case. The weighted mean matrices are considered by the author in [10].

Key Words: Transpose of Weighted Mean Matrix, Weighted Sequence Space.


1. Introduction and Notations:

In this paper, we concern with transpose of the weighted mean matrix \( A_d = (a_{n,k}) \), denoted by \( A_d^\ast \), where

\[
a_{n,k} = \begin{cases} \frac{d_k}{D_n} & \text{for } 1 \leq k \leq n \\ 0 & \text{for } k > n. \end{cases}
\]

where the \( d_n \)'s are non-negative numbers with partial sum \( D_n = d_1 + \ldots + d_n \). We insist that \( d_1 > 0 \), so that each \( D_n \) is positive.

These results are extension of some results which is considered by the author in [8] and [9] and Bennett [2] and [4]. If \( r_n = \sum_{k=1}^{n} \frac{w_k d_k}{D_k} \), and also \( R_n \) and \( W_n \) are defined as usual, then the norm of \( A_d^\ast \) on \( \ell_1(w) \) is the supremum of \( \frac{R_n}{W_n} \).

Let \( w = (w_n) \) be a decreasing, non-negative sequence with \( \lim_{n \to \infty} w_n = 0 \) and \( \sum_{n=1}^{\infty} w_n \) divergent. Write \( W_n = w_1 + \ldots + w_n \). Then \( \ell_p(w) \) (and the Lorentz sequence space \( d(w, p) \)) is the space of sequences \( x = (x_n) \) with

\[
\|x\|_{\ell_p(w)} = \left( \sum_{n=1}^{\infty} w_n |x_n|^p \right)^{1/p},
\]

convergent, where \( (x_n^\ast) \) is the decreasing rearrangement of \( |x_n| \).

We now consider the operator \( A \) defined by \( Ax = y \), where \( y_n = \sum_{k=1}^{n} a_{n,k} x_k \). We shall write \( \|A\|_{\ell_p(w)} \) for the norm of \( A \) when regarded as an operator from \( \ell_p(w) \) to \( \ell_p(w) \), where

\[
\|A\|_{\ell_p(w)} = \sup\{\|Ax\|_{\ell_p(w)} : \|x\|_{\ell_p(w)} \leq 1\},
\]

\[
\|A\|_{w,p} = \sup\{\|Ax\|_{w,p} : \|x\|_{w,p} \leq 1\}.
\]

Also, we define

\[
M_{w,p}(A) = \sup\{\|Ax\|_{\ell_p(w)} : \|x\|_{\ell_p(w)} = 1\},
\]

where \( x = (x_n) \) is regarded as a decreasing, non-negative sequences in \( \ell_p(w) \).

We assume that

1) \( a_{n,k} \geq 0 \) for all \( n, k \). This implies that \( |Ax| \leq A(|x|) \) for all \( x \), and hence the non-negative sequences \( x \) are sufficient to determine \( \|A\|_{\ell_p(w)} \).

We assume further that each \( A(e_k) \) is in \( \ell_1(w) \), that is:
Proof: By Abel summation, it follows that
\[
\sum_{k=1}^{n} a_k y_k = \sum_{k=1}^{n} a_k (Y_k - Y_{k-1}) \quad (Y_0 = 0)
\]
\[
= \sum_{k=1}^{n-1} Y_k (a_k - a_{k+1}) + a_n Y_n.
\]
Now, applying the hypothesis in both cases, we deduce that
\[
\sum_{k=1}^{n-1} Y_k (a_k - a_{k+1}) + a_n Y_n \geq \sum_{k=1}^{n} X_k (a_k - a_{k+1}) + a_n X_n = \sum_{k=1}^{n} a_k x_k.
\]
Therefore
\[
\sum_{k=1}^{n} a_k x_k \leq \sum_{k=1}^{n} a_k y_k.
\]
Corollary: Let \( x, y \) be decreasing, non-negative elements of \( \mathbb{R}^d \) (or \( \ell_1(w) \)) with \( x \ll y \).
Then
\[
\|x\|_{\ell_1(w)} \leq \|y\|_{\ell_1(w)}.
\]

Proposition 1([8], Proposition 1.4.1): Suppose that (1) holds, and that (3) for all subsets \( M, N \) of \( \mathbb{N} \) having \( m, n \) elements respectively, we have
\[
\sum_{i \in M} \sum_{j \in N} a_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}.
\]
Then
\[
\|Ax\|_{\ell_1(w)} \leq \|Ax^*\|_{\ell_1(w)} \leq \|Ax^*\|_{w,1} \leq \|Ax^*\|_{w,1}
\]
for all non-negative elements \( x \) of \( \ell_1(w)(d(w,1)) \),
where \( x^* \) is the decreasing rearrangement of \( |x| \).
Hence decreasing, non-negative sequences are sufficient to determine \( \|A\|_{\ell_1(w)}(\|A\|_{w,1}).

Proposition 2([3], Lemma 9): Let \( A = (a_{i,j})_{i,j=1}^{\infty} \) be a matrix operator with non-negative entries, and consider the associated transformation, \( x \to y \), given by \( y_i = \sum_{j=1}^{\infty} a_{i,j} x_j \).
Then the following conditions are equivalent:
(i) \( y_i \geq y_{i+1} \quad (i,n) \) follows by taking \( x \) to be the sequence \( (1,1,0,0,...) \) of \( n \) ones followed by zeros.
(ii) \( r_i, n \geq r_{i+1,n} \quad (i,n) \), and also \( (x_i) \) is decreasing, non-negative sequence, then
\[
\sum_{n=1}^{\infty} r_{i,n} (x_n - x_{n+1}) \geq \sum_{n=1}^{\infty} r_{i+1,n} (x_n - x_{n+1}) = \sum_{j=1}^{\infty} a_{i+1,j} x_j = y_{i+1}.
\]
This completes the proof of the statement.

Lemma 2: Suppose \( u = (u_n) \) and \( w = (w_n) \) are sequences of positive numbers.
(i) If \( m \leq \frac{u_n}{w_n} \leq M \) for all \( n \), then \( m \leq \frac{U_n}{W_n} \leq M \) for all \( n \).
(ii) If \( \frac{u_n}{w_n} \) is increasing (or decreasing), then so is \( \frac{U_n}{W_n} \).
(iii) If \( \frac{u_n}{w_n} \to U \) as \( n \to \infty \), then \( \frac{U_n}{W_n} \to U \) as \( n \to \infty \) (also with \( U = \infty \)).
Proof: Elementary.

3. Transpose of the Weighted Mean operator

Let now \( A_d \) be the weighted mean matrix with properties (1), (2) and (3), and \( A_d^t \) be its transpose which is defined as follows:
\[
(A_d^t x)(n) = \sum_{k=1}^{\infty} \frac{d_n x_k}{D_k}.
\]
This is an upper triangular matrix.
Recall that \((w_n)\) is said to be 1-regular if
\[
r_1(w) = \sup_{n \geq 1} \frac{W_n}{nw_n}
\]
is finite\([11]\). A pleasantly simple statement can also be made about the norm of the weighted mean matrix operator for general \(w = (w_n)\).
With the previous notation,
\[
r_n = \frac{1}{n}(w_1 + \ldots + w_n) = \frac{W_n}{n}.
\]

**Theorem 1:** Suppose \(A^t_d\) is a weighted mean operator defined as before and also \(d = (d_n)\) is such that \(nd_n \leq D_k\) \((\forall k \geq n)\). If \(w = (w_n)\) is 1-regular, then for \(p > 1\), we have:
\[
\|A^t_d\|_{w,p} \leq pr_1(w) \quad \text{and} \quad \left(\|A^t_d\|_{\ell_p(w)} \leq pr_1(w)\right).
\]

**Proof:** As mentioned before, it is sufficient to consider decreasing, non-negative sequences.
Let \(x\) be in \(\ell_p(w)\) or \(d(w,p)\) such that \(x_1 \geq x_2 \geq \ldots \geq 0\), and so \(\|x\|_{w,p} = \|x\|_{\ell_p(w)}\) and also the same is true for the norm of \(A^t_d\). Then applying \([11]\), Theorem 4.1.6, we deduce that:
\[
\|A^t_d\|_{w,p}^p = \sum_{n=1}^{\infty} w_n \left(\sum_{k=n}^{\infty} \frac{d_nx_k}{D_k}\right)^p
\]
\[
\leq \sum_{n=1}^{\infty} w_n \left(\sum_{k=n}^{\infty} \frac{x_k}{k}\right)^p
\]
\[
\leq (pr_1(w))^p \sum_{n=1}^{\infty} w_n x_n^p.
\]
\[
= (pr_1(w))^p \|x\|_{w,p}^p.
\]
Hence \(\|A^t_d\|_{w,p} \leq pr_1(w)\). This completes the proof.

**Theorem 2:** Suppose \(A^t_d\) is an operator on \(\ell_1(w)\). If
\[
R = \sup_{n} \frac{R_n}{W_n} < \infty,
\]
where \(r_n = \sum_{k=1}^{n} \frac{w_kd_k}{D_n}\), and \(R_n = r_1 + \ldots + r_n\) and \(W_n\) as usual. Then \(A^t_d\) is a bounded operator from \(\ell_1(w)\) into itself, and we have \(M_{w,1}(A^t_d) = R = \|A^t_d\|_{w,1}\).

**Proof:** Since \((A^t_d x)(n) \leq (A^t_d x^*)(n)\) for all \(n\), it is sufficient to consider decreasing, non-negative sequences. Let \(x\) be in \(\ell_1(w)\) such that \(x_1 \geq x_2 \geq \ldots \geq 0\). Then
\[
\|A^t_dx\|_{w,1} = \|A^t_dx\|_{\ell_1(w)} = \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{d_nx_k}{D_k}\right)
\]
\[
= \sum_{n=1}^{\infty} r_nx_n
\]
\[
= \sum_{n=1}^{\infty} R_n(x_n - x_{n+1})
\]
\[
\leq R \sum_{n=1}^{\infty} W_n(x_n - x_{n+1})
\]
\[
= R \sum_{n=1}^{\infty} w_nx_n.
\]
Hence \(\|A^t_d\|_{w,1} = M_{w,1}(A^t_d) \leq R\).
We have to show that this constant is the best possible. We take \(x_1 = x_1 = \ldots = x_n = 1\) and \(x_k = 0\) for all \(k \geq n + 1\). Then
\[
\|x\|_{\ell_1(w)} = W_n \quad \text{and} \quad \|A^t_dx\|_{\ell_1(w)} = R_n.
\]
Therefore \(M_{w,1}(A^t_d) = R = \|A^t_d\|_{w,1}\).

**3. Copson Operator on Weighted Sequence Spaces:**

We now consider the Copson operator \(C\) on \(\ell_1(w)\) and \(d(w,1)\), which is defined by \(y = Cx\), where
\[
y_i = \sum_{j=1}^{\infty} \frac{x_j}{j}.
\]
It is given by the transpose of the matrix of the Averaging operator \(A\):
\[
a_{i,j} = \begin{cases} 
\frac{1}{j} & \text{for } i \leq j \\
0 & \text{for } i > j 
\end{cases}
\]
This is an upper triangular matrix. The classical inequality of Copson \([5]\) and \([6]\) states that \(\|C\|_p = \|A^t_d\|_p = p(p > 1)\) as an operator on \(\ell_p\) spaces.

**Proposition 3:** If \(w = (w_n)\) is 1-regular, then \(C\) maps \(\ell_1(w)\) into \(\ell_1(w)\). Also, we have
\[
\|C\|_{w,1} = M_{w,1}(C) \leq \|C\|_{\ell_1(w)} \leq r_1(w).
\]

**Proof:** Since
\[
r_n = \frac{W_n}{n} \leq r_1(w)w_n \quad (\forall n),
\]
then by Lemma 2(i) and Theorem 2, it follows that
\[ \|C\|_{w,1} = M_{w,1}(C) \leq \|C\|_{\ell_1(w)} \leq r_1(w). \]

**Corollary 1** ([8], Theorem 2.3.1): If
\[ \sup \frac{1}{W_n} \sum_{n=1}^{n} \frac{W_k}{k} < \infty, \]
then the Copson operator is a bounded operator from \( d(w,1) \) into itself, and also we have
\[ M_{w,1}(C) = \|C\|_{w,1} = \sup \frac{1}{W_n} \sum_{n=1}^{n} \frac{W_k}{k}. \]

Write
\[ u_n = \frac{1}{r^n}, \quad (r > 0) \quad v_n = \int_{n-1}^{n} \frac{1}{t^r} dt \]
and (as usual) \( U_n = u_1 + \ldots + u_n \), etc. For \( r < 1 \), the usual integral comparison gives
\[ v_2 + \ldots + v_n \leq U_n \leq V_n, \]
or
\[ \frac{1}{1 - r} (n^{1-r} - 1) \leq U_n \leq n^{1-r}, \]
we need to know that \( \frac{U_n}{V_n} \) is increasing. The following is the key lemma.

**Lemma 3:** With \( v_n \) as above (for any \( r > 0 \)), \( n r v_n \) decreases with \( n \) and \( n^r v_{n+1} \) increases with \( n \).

**Proof:** Write \( t_n = n^r v_n \). Then
\[ t_{n+1} = (n+1)^r \int_{n}^{n+1} \frac{ds}{s^p} = (n+1)^r \int_{n-1}^{n} \frac{ds}{(s+1)^r}. \]

For \( n - 1 \leq s \leq n \), we have \( \frac{n+1}{n} \leq \frac{s+1}{s} \), hence
\[ \frac{n+1}{s^p} \leq \frac{n^r}{(s+1)^p}. \]
Therefore \( t_{n+1} \leq t_n \) (\( \forall n \)). Similarly for the second statement.

**Proposition 4:** Let \( 0 < r < 1 \) and \( U_n = \sum_{j=1}^{n} \frac{1}{j^r} \). Then \( \frac{U_n}{V_n} \) increases and tends to \( \frac{1}{1-r} \).

**Proof:** With \( v_n \) as above, by Lemma 3, \( \frac{v_n}{v_n} \) increases with \( n \), and so applying Lemma 1, we deduce that \( \frac{U_n}{V_n} \) is increasing. The limit follows from the inequalities above.

We now consider the tail of the series for \( \zeta(1+p) \). For the tail of a series, the analogous result to Lemma 2(ii) is the following.

**Lemma 4:** Suppose that \( v_n > 0 \), \( u_n > 0 \) for all \( n \) and that \( \sum_{n=1}^{\infty} v_n \) and \( \sum_{n=1}^{\infty} u_n \) are convergent. Let \( U_n = \sum_{j=n}^{\infty} u_j \), similarly \( V_n \).
\[ \left( \frac{u_n}{v_n} \right) \]
is increasing (or decreasing), then so is \( \left( \frac{U_n}{V_n} \right) \).

**Proof:** Elementary.

**Proposition 5:** Let \( r > 0 \) and let \( U_n = \sum_{j=n}^{\infty} \frac{1}{j^{1+r}} \). Then \( n^r U_n \) decreasing, \( (n-1)^r U_n \) increasing. Both tend to \( \frac{1}{r} \) as \( n \to \infty \).

**Proof:** Let \( u_n = \frac{1}{n^{1+r}} \) and \( v_n = \int_{n-1}^{n} \frac{1}{t^r} dt \). Then \( V_{n+1} = \frac{1}{rn^r} \). By the usual integral comparison,
\[ \frac{1}{rn^r} \leq U_n \leq \frac{1}{r(n-1)^r}, \]
which implies the stated limits. By Lemma 3, \( \frac{u_n}{v_n} \) is decreasing, so by Lemma 2(ii),
\[ \frac{U_n}{V_{n+1}} = rn^r U_n \]
is decreasing. Similarly, \( \frac{U_n}{V_n} \) is increasing.

**Remark:** This is stated without proof in [1], Remark 4.10.

**Theorem 3:** If \( w = \left( \frac{1}{r^n} \right), \quad 0 < p \leq 1 \), then the Copson operator \( C \) is a bounded operator from \( \ell_1(w)(d(w,1)) \) into itself. Also, we have
\[ M_{w,1}(C) = \|C\|_{w,1} = \|C\|_{\ell_1(w)} = \frac{1}{1-p}. \]

**Proof:** Since \( r_n = \frac{u_n}{v_n} \), then
\[ \frac{r_n}{w_n} = \frac{W_n}{n w_n} = \frac{W_n}{n^{1-p}}. \]
Also, since \( W_n \) is the \( U_n \) of the Proposition 4, then \( \frac{W_n}{n^{1-p}} \) increases with \( n \) and tends to \( \frac{1}{1-p} \).
Hence applying Proposition 2, we deduce that
\[ M_{w,1}(C) = \|C\|_{w,1} = \|C\|_{\ell_1(w)} = \frac{1}{1-p}. \]

**Remark:** When \( p = 1 \), so that \( w_n = \frac{1}{n} \), we have
\[ \frac{r_n}{w_n} = \frac{W_n}{n} \to \infty \quad (as \quad n \to \infty), \]
so the Copson operator \( C \) is not a bounded operator on \( d(w,1) \), although of course is satisfies condition (2).
Theorem 4: Let \( w_n \) be defined by \( W_n = n^{1-p} \), where \( 0 < p < 1 \). Then the Copson operator is a bounded operator from \( d(w, 1) \) into itself. Also, we have

\[
\| C \|_{w, 1} = M_{w, 1}(C) = \frac{1}{1 - p}.
\]

Proof: We now have

\[
R_n = \sum_{k=1}^{n} \frac{W_k}{k} = \sum_{k=1}^{n} \frac{1}{k^p}.
\]

so the new \( \frac{R_n}{W_n} \) is exactly the \( \frac{r_n}{w_n} \) of Theorem 3 and Proposition 4 again gives the statement.

4. Continuous Version of the Copson Operator:

In this section, we consider the analogous problem for the continuous case concern the space \( L_p(w) \). In the continuous case, the Copson operator \( C \) is given by:

\[
(Cf)(x) = \int_{x}^{\infty} \frac{f(t)}{t} dt.
\]

Let \( w(x) \) be a decreasing, non-negative function on \( (0, \infty) \). We assume that \( W(x) = \int_{0}^{x} w(t) dt \) is finite for each \( x \) (Hence \( \frac{1}{w} \) is permitted for \( 0 < \alpha < 1 \), but not for \( \alpha = 1 \)). Then \( L_p(w) \) is the space of functions \( f \) having

\[
\int_{0}^{\infty} w(x)|f(x)|^p dx
\]

convergent, with norm

\[
\| f \|_{L_p(w)} = \left( \int_{0}^{\infty} w(x)|f(x)|^p dx \right)^{1/p}.
\]

Proposition 6: Let \( f \geq 0 \) be in \( L_p(w) \), \( a(x) = \frac{w(x)}{W(x)} f(x) \), and also \( A_{\infty}(x) = \int_{x}^{\infty} a(t) dt \). Then \( A_{\infty}(x) \) is finite and also we have:

\[
\| A_{\infty} \|_{L_p(w)} \leq p \| f \|_{L_p(w)}.
\]

Proof: Fix \( x_0 \). For any \( x < x_0 \), let \( \int_{x}^{x_0} a(t) dt = A_{x_0}(x) \). Then \( \frac{d}{dx} A_{x_0}(x)^p = -p A_{x_0}(x)^{p-1} a(x) \), and so

\[
A_{x_0}(x)^p = A_{x_0}(x_0)^p - p \int_{x}^{x_0} A_{x_0}(t)^{p-1} a(t) dt.
\]

Hence, applying Holder’s inequality, we deduce that:

\[
\int_{0}^{x_0} w(x)A_{x_0}(x)^p dx = p \int_{0}^{x_0} w(x) \int_{x}^{x_0} A_{x_0}(t)^{p-1} a(t) dt dx = p \int_{0}^{x_0} A_{x_0}(t)^{p-1} a(t) \int_{0}^{t} w(x) dx dt = \int_{0}^{x_0} w(t) A_{x_0}(t)^{p-1} a(t) W(t) dt = p \int_{0}^{x_0} w(t) A_{x_0}(t)^{p-1} a(t) f(t) dt \leq p \left( \int_{0}^{x_0} w(t) f(t)^p dt \right)^{1/p} \left( \int_{0}^{x_0} w(t) A_{x_0}(t)^p dt \right)^{1/p'}.
\]

Therefore

\[
\left( \int_{0}^{x_0} w(t) A_{x_0}(t)^p dt \right)^{1/p} \leq p \| f \|_{L_p(w)}.
\]

The above inequality is true for all \( x_0 > 0 \), and so true with \( x_0 \) replacing by infinity. This completes the proof.

Proposition 7: If \( \frac{W(x)}{w(x)} \leq r_1(w) \) (\( \forall x > 0 \)), then

\[
\| C \|_{L_p(w)} \leq pr_1(w).
\]

Proof: We have

\[
\frac{1}{t} \leq r_1(w) \frac{w(t)}{W(t)},
\]

and so

\[
(Cf)(x) \leq r_1(w) \int_{x}^{\infty} \frac{w(t)}{W(t)} f(t) dt = r_1(w) A_{\infty}(x).
\]

This establishes the statement.

Theorem 5: If \( w(x) = \frac{1}{x^\alpha} \), where \( 0 \leq \alpha < 1 \), then

\[
\| C \|_{L_p(w)} = \frac{p}{1-\alpha}.
\]

Attained by action of \( C \) on decreasing positive functions.

Proof: (i) We have \( W(x) = x^{1-\alpha} \), and so

\[
\frac{W(x)}{x w(x)} = \frac{1}{1-\alpha} \quad (\forall x > 0).
\]

Hence

\[
\| C \|_{L_p(w)} \leq \frac{p}{1-\alpha}.
\]
(ii) Now, by taking $\varepsilon > 0$, and define $r$ by:

$$\alpha + rp = 1 + \varepsilon,$$

we deduce that:

$$f(x) = \begin{cases} \frac{1}{x^r} & \text{for } x \geq 1 \\ 1 & \text{for } 0 \leq x < 1 \end{cases}.$$

Then $f$ is decreasing and in $L_p(w)$, since $\int_0^1 \frac{1}{x^r} \, dx$ and $\int_0^\infty \frac{1}{x^{\alpha+r}} \, dx$ are convergent. Also, we have:

$$(Cf)(x) = \int_x^\infty \frac{1}{t^{r+1}} \, dt = \frac{1}{rx^r} \quad \text{for } x \geq 1,$$

and also we have:

$$(Cf)(x) \geq (Cf)(1) = \frac{1}{r} \quad \text{for } 0 < x < 1.$$

Hence $(Cf)(x) \geq \frac{1}{r} f(x) \quad (\forall x > 0)$, and so

$$\| Cf \|_{L_p(w)} \geq \frac{1}{r} \| f \|_{L_p(w)},$$

where $\frac{1}{r} = \frac{p}{1-\alpha + \varepsilon}$. Now, applying (i) and (ii) implies the statement.

References


