Dissipation Normal Form, Conservativity, Instability and Chaotic Behavior of Continuous-time Strictly Causal Systems

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Abstract: - The paper deals with structural properties of a class of strictly causal systems. It is shown that a special physically correct internal structure of a given system representation called dissipation normal form can be derived as a natural consequence of strict causality, dissipativity, minimality and asymptotic stability requirements. A proper generalization of classic Tellegen's theorem together with a concept of bi-orthonormal basis of the state velocity space have been used as basic ingredients expressing the signal energy conservation law for abstract system state space representations. It is demonstrated by examples that in continuous-time version the resulting structure represent a unifying tool for analysis and synthesis of a relatively general class of linear as well as nonlinear causal systems.

Key-Words: energy-metric function, bi-orthonormal basis, dissipation normal form, Tellegen’s principle, non-linear phenomena, instability, chaotic behavior

1 Introduction
Almost in any field of science and technology some sort of stability problem can appear. Instability and chaos are certainly the most important phenomena which should be treated before any other aspect of reality will be attacked. Hence it is not very surprising that a broad variety of approaches to the problem of stability, instability and analysis of chaotic phenomena exists. Many of the most popular techniques in the field of stability and chaos are in a certain sense related to the work of A.M.Liapunov. For instance the well known Lyapunov exponents in chaos theory or Lyapunov functions in stability theory [1, 2] can be mentioned as typical examples which seem to be energy oriented.

Tellegen’s theorem is one of the well known and appropriate forms of energy conservation statement in the field of electrical engineering [3, 4]. The most important feature of Tellegen’s approach is the fact that the energy conservation principle holds without any regard to physical nature of constituent network elements. This is the key idea of the proposed approach to problems of stability, dissipativity and chaos.

2 Abstract form of energy conservation
Certainly, any realizable system has to fulfill some causality and energy conservation requirements. Recall that existence of an abstract state space representation is necessary for a system to be causal. On the other hand causality does not imply energy conservation. In the field of electrical engineering Kirchhoff’s laws are necessary and sufficient for physical correctness of any electrical network from energy conservation point of view. Tellegen’s theorem, which is known to be one of the most powerful tools of system analysis and synthesis in electrical network theory, can be seen as a very elegant abstract form of energy conservation principle for a class physically correct system state space representations, in which voltages and currents have been chosen as state variables.

Let us briefly summarize the essential features of the original version of Tellegen’s theorem [4]. Assume that an arbitrary connected electrical network of b components is given. Let us disregard the specific nature of the network components and represent the network structure by an oriented graph with n vertices and b branches. Let the set of Kirchhoff law constraints be given in a form

\[ A_i = 0 \quad B_v = 0 \] (1)

where \( A \) is a node incidence matrix, \( B \) is loop incidence matrix, and vectors \( i \) and \( v \) are defined

\[ i = [i_1, i_2, \ldots, i_n]^T \quad v = [v_1, v_2, \ldots, v_n]^T \] (2)

Let \( J \) be the set of all vectors \( i \) and \( V \) be the set of all vectors \( v \) such that \( i \) and \( v \) satisfy (1). Both the
vectors of currents and voltages are elements of a b-dimensional vector space with the inner product. Then the Tellegen’s principle follows from:

**Theorem 1. (Classical Tellegen’s theorem - CTT)**

If \( i \in I \) and \( v \in V \) then it holds

\[
\forall t : \langle i(t), v(t) \rangle = 0 
\]

(3)

That is to say \( J \) and \( V \) are orthogonal subspaces of the Euclidean space \( E_h \). Furthermore \( J \) and \( V \) together span the vector space \( E_h \).

It is obvious fact, following directly from the definition of inner product, that relation (3) is just a form of constant energy statement for a class of representations in which elements of a set of voltages and currents have been chosen as state variables, as well as components of a gradient vector of a scalar field in the state space.

Let \( \mathcal{R}\{S\} \) is a continuous-time finite dimensional time-invariant strictly causal nonlinear system state space representation given by:

\[
\mathcal{R}\{S\} : \dot{x}(t) = f[x(t)] + Bu(t), \quad x(t_0) = x_0, 
\]

\[
y(t) = Cx(t), 
\]

(4)

The arbitrariness in the choice of state coordinates motivates introducing a group of state- and feedback-transformations on which the generalization of classical Tellegen’s principle has been proposed in [4].

\[
\exists \phi, \exists T, T^{-1} : \bar{x} = T(x), \quad \bar{u} = \phi(u, \bar{x}) : \langle f, (\text{grad} \ E) \bar{x} \rangle = 0 \iff \langle \bar{x}, \bar{x} \rangle = 0
\]

(5)

\[
\forall t : E[\bar{x}(t)]=E[x(t)] 
\]

For a class of discrete-time finite dimensional internal system representations \( \mathcal{R}\{S\} \) given by

\[
x(k+1) = f[x(k)] + w(k), \quad w(k) = Bu(k), \quad y(k) = Cx(k) 
\]

(6)

Similarly as in the case of continuous-time systems, a new discrete-time generalization of classical Tellegen’s principle has been introduced in [4]. If any input \( u(k) \) and any state value \( x(k) \) will be chosen then the next state value \( x(k+1) \) is given, and the state difference vector \( \Delta x(k) \) can be defined as

\[
\Delta x(k) = x(k+1) - x(k) = \Delta x_k, \quad k \in \{0,1,2,\ldots\}
\]

(7)

together with a row “gradient vector” \( \eta(k) \) defined by:

\[
\eta(k) = \frac{1}{2}[x(k+1) + x(k)]^T = \eta_k, \quad k \in \{0,1,2,\ldots\} 
\]

(8)

Interpretation of the vector \( \eta_k \) as a natural discrete-time energy function gradient vector is obvious, and the discrete-time generalization of Tellegen’s principle is then given by the inner product:

\[
\forall t = k, k \in \{0,1,2,\ldots\} : \langle \Delta x_k, \eta_k \rangle = 0 
\]

(9)

For deeper understanding a geometric interpretation of the generalized Tellegen’s principle is visualized at the

![Fig.1. Geometric interpretation of the generalized Tellegen’s principle](image)

(10)

(11)

3 Dissipativity and Stability

Let us consider the class of continuous-time nonlinear time-varying strictly causal systems given by the state space representation

\[
R(S) : \dot{x}(t) = f[t; x(t), u(t)] 
\]

\[
y(t) = h[t; x(t)]
\]

(10)

(11)

with \( t \) as continuous time variable,

\[ x_1, x_2, \ldots, x_n \]

\[ u_1, u_2, \ldots, u_r \]

as the state space coordinates,

\[ \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \]

as coordinates of the state velocity,

\[ y_1, y_2, \ldots, y_p \]

as the input signals, and with

\[ y_1, y_2, \ldots, y_p \]

as the observed output signals

Recall that according to Liouville’s theorem of vector analysis, dissipative systems have the important property that any volume of the state space strictly decreases under the action of the system flow. For nonlinear system representations \( \mathcal{R}\{S\} \) with the state velocity given by a nonlinear vector field \( f \) the property of dissipativity is defined by using the operation of divergence as follows [2].

**Definition 1: (Dissipativity of a vector field)**

The representation \( \mathcal{R}\{S\} \) with the state velocity vector field \( f \) is dissipative if it holds

\[
\text{div} \ f(x) = \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i} < 0 
\]

(12)

Let us now define a constituent set of finite number of non-interacting elementary subsystems

\[
\begin{align*}
\dot{x}_1 &= f_1(t, x_1, u_1), & y_1 &= h_1(t, x_1) \\
\dot{x}_2 &= f_2(t, x_2, u_2), & y_2 &= h_2(t, x_2) \\
&\quad \quad \vdots & \quad \quad \quad \quad \vdots \\
\dot{x}_n &= f_n(t, x_n, u_n), & y_n &= h_n(t, x_n)
\end{align*}
\]

(13)

It follows that the constituent set (29) is dissipative if at least one of the elementary subsystems is dissipative.

**Remark 1:** It is easy to deduce that the constituent set of non-interacting subsystems with zero input and with unique equilibrium state is locally asymptotic stable iff each of the elementary subsystems is dissipative. It means that in general dissipativity is necessary but not sufficient for asymptotic stability.
Remark 2: Nonlinear systems having at an equilibrium state a dissipative approximate linearization are locally dissipative in a neighborhood of this equilibrium state, but need not to be globally dissipative, i.e. their region of dissipation need not be the whole state space.

Remark 3: Recall that systems with
\[ \text{div } f(x) = 0 \] (14)
preserve volume along state trajectories; such systems are usually referred to as conservative. Notice that this concept of conservativity is not always compatible with the classical meaning of the term conservative as energy preserving (or Hamiltonian).

Remark 4: Notice that a linear time invariant system
\[ x(t) = Ax(t) + Bu(t), \quad x(t_0) = x^0, \] (15)
is dissipative if and only if its matrix A has negative trace, i.e. if it holds
\[ \text{Tr } A < 0 \] (16)
Thus an asymptotically stable linear system is always dissipative, while the converse is not true in general.

4 Minimality of state velocity space
It is challenging to find such a structure of interactions between the elements of the constituent set that the intrinsic relations between fundamental system properties such as dissipativity, conservativity, asymptotic stability, instability, state and parameter minimality and chaoticity will be clearly displayed. In order to achieve the aim, it is reasonable to specify the minimal dimension of the state velocity space. We start with a concept of the Hessenberg matrix.

Definition 2: (Hessenberg structure of a matrix)
Let A is a n-th order rectangular matrix. We say that the matrix A has the Hessenberg structure if it holds
\[ a_{i,j} = 0, \quad j > i + 1 \] (17)
\[ a_{i,i+1} \neq 0, \quad \text{and} \quad \text{sign}(a_{i,i+1}) = 1 \] (18)

Definition 3: (Hessenberg structure of a vector field)
A vector field \( f \) has the Hessenberg structure if it holds
\[ \frac{\partial f_j}{\partial x_i} = 0, \quad j > i + 1 \] (19)
\[ \frac{\partial f_{i+1}}{\partial x_i} \neq 0, \quad \text{sign} \left( \frac{\partial f_{i+1}}{\partial x_i} \right) = 1 \] (20)

Let a n-th order system representation is given
\[ \Re\{S\} : \dot{x}(t) = f[x(t)] + Bu(t), \quad x(t_0) = x^0, \] (21)
and the matrices B and C have the form
\[ C = [c_1, 0, \ldots, 0], \quad B^T = [0, 0, \ldots, b_n] \] (22)

Definition 4: (Hessenberg structure of a system)
We say that a system representation (15),(16) has the Generalized Hessenberg structure if vector field \( f \) has the Hessenberg structure
\[ \frac{\partial f_j}{\partial x_i} = 0, \quad j > i + 1 \] (23)
\[ \frac{\partial f_{i+1}}{\partial x_i} \neq 0, \quad \text{sign} \left( \frac{\partial f_{i+1}}{\partial x_i} \right) = 1 \] (24)
and in addition if it holds
\[ c_i \triangleq \frac{\partial h}{\partial x_i} \neq 0, \quad \text{sign} \left( \frac{\partial h}{\partial x_i} \right) = 1 \] (25)
\[ b_n \triangleq \frac{\partial f_n}{\partial u_n} \neq 0, \quad \text{sign} \left( \frac{\partial f_n}{\partial u_n} \right) = 1 \] (26)

Remark 5: It is worthwhile to notice that each of the Jacobian matrices \( J_x(f), J_u(f), J_x(h) \) has a properly defined structure motivated by the system structure corresponding to the cascade connection of the elementary subsystems according to the Fig.2.

![Fig.2. Generalized Hessenberg structure](image)

For the internal structure of subsystems \( S_k \) see Fig.3.

![Fig.3. Internal structure of the elementary subsystem](image)
5 Bi-orthonormal basis of velocity space

In order to specify the physically correct internal system structure in the sense of energy conservation principle validity [4], we introduce a structural representation

\[ R^* \{ S \} : \quad Q \dot{x}(t) = A^* x(t) + B^* u(t) \quad (32) \]

\[ y(t) = C^* x(t) \]

Let us assume that each elementary subsystem \( S_i \) of the constituent set is dissipative, i.e. it holds

\[ \forall i: \quad \frac{\partial f_i}{\partial x_i} < 0, \quad i = 1, 2, \ldots, n \quad (33) \]

Then the simplest form of the structural matrix \( A^* \) in the Generalized Hessenberg representation reads

\[ A^* = \begin{bmatrix}
-1 & 1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 1 & \ldots & 0 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix} \quad (34) \]

Now, let the structural matrices \( Q, B^*, C^* \) be given by

\[ Q = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & \ldots & 0 & 0 & 0
\end{bmatrix}, \quad B^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad (C^*)^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (35) \]

where the columns \( q_1, q_2, \ldots, q_n \) of the matrix \( Q \) form a biorthonormal basis in the state velocity space given by

\[ q_k + q_{k+1} = e_k, \quad k = 1, 2, \ldots, n-1 \]

\[ q_n = e_n \quad (36) \]

Because \( Q \) is always invertible, we have

\[ Q^{-1} = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & 1
\end{bmatrix} \quad (37) \]

and a resulting generic structure of the matrix \( A \) follows

\[ A = Q^{-1} A^* = \begin{bmatrix}
-1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & -1 & 0
\end{bmatrix} \quad (38) \]

6 Structure of continuous-time systems in dissipation normal form

Our goal is to specify a class of strictly causal system representations for which a form of energy conservation such as the Generalized Tellegen’s principle holds. We start with the hypothesis that it is not the physical energy by itself, but only a measure of distance from the system equilibrium to the actual state \( x(t) \), what is needed for this aim. Thus, instead of the physical energy a metric \( \rho[x(t), x^*] \) will be defined in a proper way, and for an abstract energy \( E(x) \) we then put formally:

\[ E(x) = \frac{1}{2} \rho^2 \left[ x(t), x^* \right] = \frac{1}{2} \| x(t) - x^* \|^2 \quad (39) \]

It has been shown in [1], [3], that the resulting state equivalent system representation in dissipation normal form, corresponding to the derived generic structure (35), (38) is described by a triple of matrices \( \{ A, B, C \} \) as follows

\[ \bar{A} = \begin{bmatrix}
-\alpha_1 & \alpha_2 & 0 & 0 & \ldots & 0 & 0 \\
-\alpha_2 & 0 & \alpha_3 & 0 & \ldots & 0 & 0 \\
0 & -\alpha_3 & 0 & \alpha_4 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -\alpha_{n-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -\alpha_n
\end{bmatrix} \quad (40) \]

\[ \beta = \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\vdots \\
\beta_{n-1} \\
\beta_n
\end{bmatrix} \quad (41) \]

It is easy to show that the set of real basic design parameters \( \alpha_i, \gamma, \beta_i \) must satisfy the following fundamental consistency conditions:

1. \( \forall i, \quad i \in \{1, 2, \ldots, n\} : 0 < \alpha_i < \infty \Leftrightarrow \) for structural asymptotic stability
2. \( \forall i, i \in \{2, 3, \ldots, n\} : 0 \neq \alpha_i, \gamma \neq 0, \exists i : \beta_i \neq 0 \Leftrightarrow \) for structural minimality

The generic internal structure of an \( n \)-th order continuous-time strictly causal system in dissipation normal form is shown at the Fig. 4.

![Fig. 4. Internal structure of continuous-time strictly causal system in the dissipation normal form](image-url)


7 Dissipativity and Stability Analysis

Example 1. (Stability analysis of a linear system)

Let the n-th order system representation is given by the linear differential equation with constant coefficients

\[ y^{(6)} + a_1 y^{(5)} + \ldots + a_5 y(t) + a_6 y(t) = 0 \]  (41)

with characteristic polynomial

\[ P(s) = s^6 + a_1 s^5 + a_2 s^4 + \ldots + a_5 s + a_6 \]

and with matrix \( A \) in the dissipation normal form

\[ A = \begin{bmatrix}
-\alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 \\
-\alpha_2 & 0 & \alpha_3 & 0 & 0 & 0 \\
0 & -\alpha_3 & 0 & \alpha_4 & 0 & 0 \\
0 & 0 & -\alpha_4 & 0 & \alpha_5 & 0 \\
0 & 0 & 0 & -\alpha_5 & 0 & \alpha_6 \\
0 & 0 & 0 & 0 & -\alpha_6 & 0 \\
\end{bmatrix} \]  (42)

Hence the parameters \( a_i, \ i \in \{1, 2, \ldots, 6\} \) are given by

\[
\begin{align*}
a_1 &= \alpha_1 \\
a_2 &= \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 \\
a_3 &= \alpha_1 \left( \alpha_2^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 \right) \\
a_4 &= \alpha_2 \left( \alpha_2^2 + \alpha_5^2 + \alpha_6^2 \right) + \alpha_3 \left( \alpha_3^2 + \alpha_6^2 \right) + \alpha_4 \alpha_5^2 + \alpha_6^2 \\
a_5 &= \alpha_1 \alpha_2 \left( \alpha_2^2 + \alpha_6^2 \right) + \alpha_1 \alpha_3 \alpha_4^2 \\
a_6 &= \alpha_2^2 \alpha_5^2 + \alpha_6^2 \\
\end{align*}
\]

Recall that the necessary and sufficient condition for existence of the unique equilibrium state \( x^* = 0 \) is

\[ \det A = a_6 = \alpha_2^2 \alpha_5^2 \alpha_6^2 \neq 0 \]  (43)

From the existence of a unique equilibrium state point of view, the dissipation parameter \( \alpha_1 \), as well as interaction parameters \( \alpha_3, \alpha_5 \) can be chosen arbitrary. Now, let all the parameters \( \alpha_1, \alpha_2, \ldots, \alpha_n \) of \( P_n(s) \) be considered as unknown, and let us specify the region of asymptotic stability in the parameter space.

The representation in dissipation normal form reads

\[
\begin{align*}
\dot{x}_1(t) &= -\alpha_1 x_1(t) + \alpha_2 x_2(t) \\
\dot{x}_2(t) &= -\alpha_2 x_2(t) + \alpha_3 x_3(t) \\
\dot{x}_3(t) &= -\alpha_3 x_3(t) + \alpha_4 x_4(t) \\
\dot{x}_4(t) &= -\alpha_4 x_4(t) + \alpha_5 x_5(t) \\
\dot{x}_5(t) &= -\alpha_5 x_5(t) + \alpha_6 x_6(t) \\
\dot{x}_6(t) &= -\alpha_6 x_6(t) \\
y(t) &= \gamma x_1(t)
\end{align*}
\]  (44)

and for the Euclidean metric \( \rho = \rho_2 \)

\[
E[x(t)] = \frac{1}{2} \rho^2 |x(t)| = \frac{1}{2} \rho^2 x(t)^2 = \frac{1}{2} \sum_{i=1}^{6} x_i^2(t),
\]  (45)

it holds

\[ l^* \ E(x) = 0 \iff x(t) = x^*, \ (x^* = 0) \]

\[ 2^* \ x_i(t) \in R \iff x_i^2(t) \geq 0 \implies E(x) > 0 \iff x(t) \neq x^* \]

For the derivative of the signal energy function \( E(x) \) along the system representation (44) we get

\[ \frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\alpha_1 x_1^2(t) = -\frac{\alpha_1}{\gamma^2} y^2(t) \]  (46)

where \( \gamma \) is a real output scaling parameter

\[ 0 < \gamma < \infty \]  (47)

Thus, for non-zero output dissipation power \( y^2(t) \) the signal energy conservation principle holds if and only if:

\[ P(t) = y^2(t) \iff \alpha_1 = \gamma^2 > 0 \]  (48)

Remark 2: Notice that the dissipativity parameter \( \alpha_1 \) is the only element of the matrix \( A \), which sign separates the system dissipativity from its anti-dissipativity. The critical value of \( \alpha_1 = 0 \), corresponds to the system conservativity and separates stability of the equilibrium state from its anti-stability.

Remark 3: Notice, that if we put \( \alpha_3 = 0 \), then the state variables \( x_i, \ i = 5,6 \) become unobservable by the output \( y \); thus only the first isolated subsystem with the state variables \( x_i, \ i = 1,2,3,4, \) which is observable, will be asymptotic stable, while the second one will oscillate on the constant energy level, (see Fig.3.c for energy evolution). Similarly, if we put \( \alpha_5 = 0 \), then the state variables \( x_i, \ i = 3,4,5,6 \) become unobservable by the output \( y \), and only the observable subsystem

\[
\begin{align*}
\dot{x}_1(t) &= -\alpha_1 x_1(t) + \alpha_2 x_2(t) \\
\dot{x}_2(t) &= -\alpha_2 x_2(t) + \alpha_4 x_4(t) \\
\dot{x}_4(t) &= -\alpha_4 x_4(t) + \alpha_6 x_6(t) \\
y(t) &= \gamma x_1(t)
\end{align*}
\]  (49)

will be asymptotic stable (see Fig.3b)

\[ \text{Fig. 5. Time evolution of the signal energy } E[x(t)] \]

a) conservative case \( \alpha_1 = 0, \alpha_k \)-arbitrary for \( k = 2,3,\ldots,n \)

b) stability \( \alpha_1 > 0, \alpha_2 = 0, \)

c) stability \( \alpha_1 > 0, \alpha_5 = 0, \)

d) asymptotic stability \( \alpha_1 > 0, \alpha_k \neq 0, \) for \( k = 2,3,\ldots,n \)
8 Dissipativity and nonlinear phenomena

Example 2. (Non-linear stability analysis)

Let us consider a simple non-linear system given by
\[
\ddot{y}(t) + \varepsilon\left[\alpha - \beta y^2(t)\right] \dot{y}(t) + a_2y(t) = 0 \tag{50}
\]
If \( C \) is defined by \( C = [\gamma, 0] \), and \( A(x) \) is defined by the non-linear dissipation normal form
\[
A(x_1, x_2) = \begin{bmatrix}
-\varepsilon\left[\alpha - \frac{1}{\gamma}\beta x_1^2\right], & \sqrt{a_2} \\
-\sqrt{a_2}, & 0
\end{bmatrix} \tag{51}
\]
then the system representation is locally observable if \( \gamma > 0, a_2 > 0 \) \( \tag{52} \)
and the signal energy conservation principle gives
\[
\frac{dE(t)}{dt}_{\mathbb{R}_0(\varepsilon)} = -P \leq 0, \quad P = \varepsilon\left[\alpha - \frac{1}{2}\beta x_1^2\right]x_1^2 \tag{53}
\]
It follows that the unique equilibrium state \( x^* = 0 \) is asymptotically stable in the region \( D \subset X \subset \mathbb{R}^2 \)
\[
D = \left\{ x_1, x_2 : |x_1| < \sqrt{\frac{3\alpha}{\beta}} \right. \text{ and } x_1^2 + x_2^2 < \frac{3\alpha}{\beta} \} \tag{54}
\]
if \( \varepsilon > 0, \alpha > 0, \beta > 0, a_2 > 0 \).

Example 3. (Generation of Lyapunov functions)

Let the same non-linear system be given
\[
\ddot{y}(t) + \varepsilon\left[\alpha - \beta y^2(t)\right] \dot{y}(t) + a_2y(t) = 0 \tag{55}
\]
but instead of the dissipation normal form the state vector \( x(t) \) is defined by
\[
x_1 = y, x_2 = dy/dt \tag{56}
\]
Then the corresponding system representation is structurally observable with the observability matrix \( H_x = I \), and from the signal energy conservation principle
\[
\frac{dV(t)}{dt}_{\mathbb{R}_0(\varepsilon)} = -P \leq 0, \quad P = \varepsilon\left[\alpha - \frac{1}{2}\beta x_1^2\right]x_1^2 \tag{57}
\]
a unique Lyapunov function \( V(x) \) can be determined by isometric transformations. For \( \alpha = \beta = a_2 = 1 \) we get
\[
V(x) = \frac{1}{2}\left[ x_1^2 - 2x_1^2 + (1 + \varepsilon^2)x_1^2 - 2\varepsilon x_1x_2 + 2\varepsilon x_1x_2 + x_2^2 \right] \tag{58}
\]
and for linear conservative case (\( \varepsilon = 0 \)) it reduces to
\[
V(x) = \frac{1}{2}\left[ x_1^2 + x_2^2 \right] \tag{59}
\]

Example 4. (Generation of chaos in a causal system)

Let a 4th order system represented by dissipation normal form with chaotic state be given by
\[
\begin{align*}
\dot{x}_1 &= -\alpha_1x_1 + \alpha_2x_2 \\
\dot{x}_2 &= -\alpha_3x_2 + \alpha_4x_3 \\
\dot{x}_3 &= -\alpha_4x_3 + \alpha_5x_4 \\
\dot{x}_4 &= -\alpha_3x_1
\end{align*}
\]
\[
\alpha = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{bmatrix} = \begin{bmatrix}
-1 + 10x_1^2 \\
1 \\
1 \\
2.00
\end{bmatrix}, \quad \varepsilon = 0.5 \tag{60}
\]

9 Conclusion

In the present paper basic concepts concerning dissipativity, conservativity, state minimality, internal stability, instability and chaos have been examined from a unified structural point of view. Both the linear as well as non-linear state-output system representations are discussed.

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References: